PACS File #2510

# Economic Stabilization Policy: A Survey

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September 1981

Staff Report #68

This paper reviews selected studies in the theory of macroeconomic stabilization policy and summarizes their key findings. A simple model is constructed which includes all surveyed models as special cases. All solutions are derived and described step by step.

The views expressed herein are soley those of the author and do not necessarily represent the views of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

# ECONOMIC STABILIZATION POLICY: A SURVEY

## Introduction

This paper reviews selected studies in the theory of macroeconomic stabilization policy and summarizes their key findings. Although this theory still is relatively new, much has already been learned. It is the purpose of this survey, then, to make the existing body of knowledge more accessible to policymakers and students alike by consolidating it and presenting it in a unified fashion.[1/]

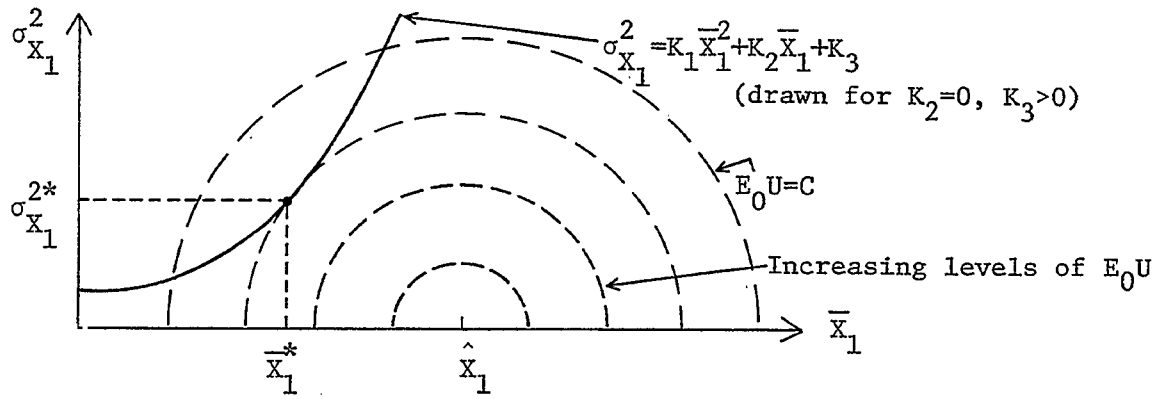
The problem generally posed in stabilization theory is to determine values of policy variables which maximize an objective function subject to laws governing the economy's motion. The arguments of the objective function, called goal variables, are assumed to be certain aggregate variables, such as the unemployment rate and inflation rate, and the motion of the economy is represented by a stochastic difference equation. Optimal policy is then derived as a rule which describes how policy variables should be set based on current information.

The validity of this seemingly straightforward approach to the policymaking problem has been challenged based on recent developments in general equilibrium theory. Robert E. Lucas, Jr., (1972) has constructed a general equilibrium model of the business cycle, and Neil Wallace (1980) has augmented that model to allow open market operations. These models suggest that economic stabilization policy may not be desirable.

In general equilibrium models individuals optimize and markets clear. Policies can be evaluated according to the Pareto criterion, so there is no need to posit an arbitrary policymaker objective function. And what is damaging is that active stabilization policies do not in general yield Pareto optimal outcomes in these models.

Business cycles emerge in general equilibrium models by assuming the

Figure 6




In the random shocks model $\sigma_2^2$ and $\sigma_{23}$ are zero so that the frontier to the mean-variance opportunity set is simply the horizontal line $\sigma^2_{X_1} = \sigma_3^2$. The tangency of the frontier and indifference curve is then at $\bar{X}_1 = \hat{X}_1$. As the variance of $\theta_2$ grows, the frontier becomes steeper and the tangency point moves further away from $\hat{X}_1$.

Maximum utility is found by substituting the expression for $\tilde{P}_1$ into (iii) which yields

$$E_0\tilde{U} = -\left(\frac{\sigma_2^2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3)^2 - 2\left(\frac{\bar{\theta}_2\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3)$$

$$+\left(\frac{\sigma_{23}^2}{\sigma_2^2+\bar{\theta}_2^2} - \sigma_3^2\right).$$

When $\sigma_2^2 = 0$, the expression indicates that $E_0\tilde{U} = -\sigma_3^2$, which is the negative of the irreducible variance of $X_1$. When $\sigma_2^2 > 0$ and $\sigma_{23} = 0$, $E_0\tilde{U}$ is the sum of the first term and $-\sigma_3^2$, where the first term measures the expected loss from not closing the goal variable gap entirely.

Let us now turn to the two-period case.

2. Two-period horizon ($\gamma \neq 0$).

The problem is

$$(1) \qquad \max_{P_1, P_2} \; -E_0[(X_1-\hat{X}_1)^2+\gamma E_1(X_2-\hat{X}_2)^2]$$

subject to

$$(2) \qquad X_1 = \bar{\theta}_1 X_0 + \theta_2(1)\cdot P_1 + \theta_3(1)$$

$$X_2 = \bar{\theta}_1 X_1 + \theta_2(2)\cdot P_2 + \theta_3(2)$$

where by (2a) and (2b) for s=0, 1 and t>s

$$E_s[\theta_2(t),\theta_3(t)] = (\bar{\theta}_2,\bar{\theta}_3)$$

and

$$E_s\begin{pmatrix}\theta_2(t)-\bar{\theta}_2 \\ \theta_3(t)-\bar{\theta}_3\end{pmatrix}(\theta_2(t')-\bar{\theta}_2,\theta_3(t')-\bar{\theta}_3)$$

$$= \begin{cases} 0_{3x3} & t\neq t' \\ \begin{pmatrix}\sigma_2^2 & \sigma_{23} \\ \sigma_{23} & \sigma_3^2\end{pmatrix} & t=t' \end{cases}$$

and (3) $\underline{X}_1 = X_1$.

By the Bellman principle $\tilde{P}_2$ is found by maximizing $-\gamma E_1(X_2-\hat{X}_2)^2$ with respect to $P_2$ subject to $X_2 = \bar{\theta}_1 X_1 + \theta_2(2)P_2 + \theta_3(2)$. As was the case in the random shocks model, this problem is precisely the same as the one-period problem with the subscripts moved up one period. Thus, it follows

$$\tilde{P}_2 = \left(\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3) - \left(\frac{\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right)$$

and with $\sigma_{X_2}^2$ and $E_1X_2$ conditional on $\tilde{P}_2$,

$$\tilde{U}_2(I_1) \equiv -\gamma[\sigma_{X_2}^2+(E_1X_2-\hat{X}_2)^2]$$

$$= -\gamma[(1-H)(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)^2+2HK(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)+(\sigma_3^2-HK^2)],$$

where

$$H \equiv \frac{\bar{\theta}_2^2}{\sigma_2^2+\bar{\theta}_2^2}, \quad K \equiv \frac{\sigma_{23}}{\bar{\theta}_2}.$$

Optimal first-period policy is found by maximizing with respect to $P_1$

$$E_0U = -E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1)$$

subject to

$$X_1 = \bar{\theta}_1X_0+ \theta_2(1)P_1 + \theta_3(1)$$

and

$$\tilde{U}_2(I_1) = -\gamma[(1-H)(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)^2+2HK(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)+(\sigma_3^2-HK^2)].$$

As we found in the one-period case (expression (iii))

$$-E_0(X_1-\hat{X}_1)^2 = -\sigma_2^2P_1^2 - 2\sigma_{23}P_1 - \sigma_3^2 - (\bar{\theta}_1X_0+\bar{\theta}_2P_1+\bar{\theta}_3-\hat{X}_1)^2.$$

After some calculations and manipulations we find

$$E_0\tilde{U}_2(I_1) = -\gamma(a_0+2a_1P_1+a_2P_1^2)$$

where

$$a_0 = (1-H)[(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_3)^2-2\bar{\theta}_1\bar{\theta}_3(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_3)$$

$$+ \bar{\theta}_1^2(\sigma_3^2+\bar{\theta}_3^2)] + 2HK(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_1\bar{\theta}_3-\bar{\theta}_3) + (\sigma_3^2-HK^2),$$

$$a_1 = \bar{\theta}_1\{(1-H)[\bar{\theta}_1(\sigma_{23}+\bar{\theta}_2\bar{\theta}_3)-\bar{\theta}_2(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_3)]-HK\bar{\theta}_2\}$$

$$a_2 = (1-H)\bar{\theta}_1^2(\sigma_2^2+\bar{\theta}_2^2).$$

Differentiating $E_0U$ with respect to $P_1$ and setting the resulting expression equal to zero yields

$$\tilde{P}_1 = \left(\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2}\right)\left[\frac{\hat{X}_1+\gamma(1-H)\bar{\theta}_1(\hat{X}_2-\bar{\theta}_3)}{1+\gamma(1-H)\bar{\theta}_1^2} - \bar{\theta}_1X_0 - \bar{\theta}_3\right]$$

$$- \left(\frac{\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right)\left[1 - \frac{\gamma H\bar{\theta}_1}{1+\gamma(1-H)\bar{\theta}_1^2}\right].$$

It is interesting to compare the expression above with the expression for optimal policy in the one-period case. In general they are not the same. Thus, even with as many policy control variables as goal variables, policy cannot be made myopically when there is uncertainty about the effects of policy.

In our example optimal policy can be made myopically if $\hat{X}_2 = \bar{\theta}_1\hat{X}_1 + \bar{\theta}_3$ and $\sigma_{23} = 0$. There is no apparent reason why either of these equalities should hold. Suppose, however, that $\sigma_{23} = 0$. It then follows with $\bar{\theta}_2 > 0$ that the best myopic policy $P_1^m$ is related to the best overall policy by $P_1^m \gtreqless \tilde{P}_1$ as $\hat{X}_2 \lesseqgtr \bar{\theta}_1\hat{X}_1 + \bar{\theta}_3$. Thus, the best myopic policy implicitly assumes that the dynamic structure of the economy is optimal: once the goal variable is on its target path, it assumes the economic process will keep it there. If the target for the goal variable in the future period is above (below) what the economic process could be expected to produce, the myopic policy will be too restrictive (expansionary) in the current period.

In general, policy cannot be made myopically because what is attainable in the second period depends on the value of the policy control variable in the first period. To ignore this dependence results in a loss in expected utility. From the one-period case we know that the mean-variance frontier in the second period conditional on $X_1$ can be written

$$\sigma^2_{X_2} = (\alpha^2\sigma_2^2)\cdot\bar{X}_2^2 + 2\alpha(\beta\sigma_2^2+\sigma_{23})\cdot\bar{X}_2 + (\beta^2\sigma_2^2+2\beta\sigma_{23}+\sigma_3^2),$$

where

$$\sigma^2_{X_2} = E_1(X_2-\bar{X}_2)^2$$

$$\bar{X}_2 = \bar{\theta}_1X_1 + \bar{\theta}_2P_2 + \bar{\theta}_3$$

$$\alpha \equiv \frac{1}{\bar{\theta}_2}$$

and

$$\beta \equiv -\frac{(\bar{\theta}_1X_1+\bar{\theta}_3)}{\bar{\theta}_2}.$$

The location of the frontier in the mean-variance plane depends on the level of $X_1$ as is seen in the expressions for $\bar{X}_2$ and $\beta$. The level of $X_1$, meanwhile, depends on $P_1$, although this dependence is random as of time zero. That is, the precise relationship which obtains between $X_1$ and $P_1$ depends on the realizations of $\theta_2(1)$ and $\theta_3(1)$. It is clear, however, that the location of the mean-variance frontier for $X_2$ which is expected conditional on $I_0$ depends on $P_1$. In order to maximize expected utility conditional on $I_0$, it is necessary, therefore, to recognize how the choice of $P_1$ restricts future policy choices.

B. Long and Variable Policy Lags

Milton Friedman (1969) argued that feedback policies are likely to be destabilizing when there are long and variable policy lags. There are at least

two interpretations of his argument. One is that he is referring to the likelihood of a policy drawn at random being destabilizing, while a second is that he is referring to an optimal policy.

Under the first interpretation, the argument is that because of insufficient knowledge or institutional constraints, feedback policies in practice are likely to deviate substantially from optimal policies.[19/] Since long and variable lags add variability to the economic structure, they then increase the likelihood that policies in practice will be destabilizing. If this is Friedman's argument, he is undoubtedly correct: the more variable the economic structure, the more likely a policy chosen at random will be destabilizing.

Under the second interpretation, the argument is that with sufficient variability in the economic structure even the best feedback policy will be destabilizing. The optimal policy then must be a nonfeedback rule, such as a constant growth rate for the Fed's portfolio. In this section we will examine the validity of this argument in a simple two-period version of a model by Fischer and Cooper (1973).

Fischer and Cooper first defined in the context of a particular model the notions "length of lag" and "variability of lag." They then used numerical solution techniques to evaluate different policies. They concluded that as long as the variances of lag weights are finite, feedback rules still dominate nonfeedback rules. However, the greater the variances in lag weights, the less policymakers should respond to new information.

Fischer and Cooper assumed an infinite horizon problem with lag weights on policy from a specific family of lag distributions:

$$\max_{P} E_0 U \equiv \max_{P_1, P_2, \ldots} E_0 \sum_{t=1}^{\infty} (X_t - \hat{X})^2$$

subject to

$$X_t = \beta X_{t-1} + \sum_{i=0}^{\infty} w_i(t) \cdot P_{t-i} + U(t)$$

where $\beta$ is a known scalar and U is a random residual. They then defined the length of lag and the variability of lag with respect to the assumed family of lag distributions.[20/] While the model could be solved explicitly in the known lag case, it could not in the variable lag case.

The problems of definition and solution faced by Fischer and Cooper are not encountered in our two-period treatment of their model. We can express the model as:

(1) $$\max_{P_1, P_2} E_0 U = -E_0[(X_1 - \hat{X}_1)^2 + \gamma E_1 (X_2 - \hat{X}_2)^2]$$

subject to

(2) $$X_1 = \theta_1(1) X_0 + \theta_2(1) P_1 + \theta_3(1)$$
$$X_2 = \theta_1(2) X_1 + \theta_2(2) P_2 + \lambda P_1 + \theta_3(2)$$

where

(2a) $$\overline{\theta(t)}_s = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3) \quad t>s$$

(2b) $$\sum(t,t')_s = \begin{cases} 0_{3x3} & t \neq t' \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix}, \sigma_2^2, \sigma_3^2 > 0 & t=t'>s, \end{cases}$$

$\lambda$ has mean $\bar{\lambda}$, variance $\sigma_\lambda^2$, and is uncorrelated with the $\theta$'s, and

(3) $$\underline{X}_1 = X_1.$$

In order to make this model comparable to the one in IIA, we must substitute for $P_1$ in the second equation of (2) using the expression in the first equation of (2):

$$X_2 = \theta_1(2)X_1 + \theta_2(2)P_2 + \lambda\frac{(X_1-\theta_1(1)X_0-\theta_3(1))}{\theta_2(1)} + \theta_3(2)$$

or

$$X_2 = [\theta_1(2) + \frac{\lambda}{\theta_2(1)}]X_1 + \theta_2(2)P_2 + [\theta_3(2) - \lambda\frac{(\theta_1(1)X_0+\theta_3(1))}{\theta_2(1)}]$$

$$\equiv \theta_1'(2)X_1 + \theta_2'(2)P_2 + \theta_3'(2).$$

When we now compare this model to the one in IIA, we find two important differences. In general with $\bar{\lambda} \neq 0$ and $\sigma_\lambda^2 > 0$

(i) There is serial correlation in the residuals, $E_0\theta_3(1)\theta_3'(2) \neq 0$ and

(ii) The system lag $\theta_1'(2)$ becomes random even if $\theta_1(1)$ and $\theta_2(1)$ are known.

We are interested primarily in how the policy rule for $\tilde{P}_1$ changes as $\bar{\lambda}$ and $\sigma_\lambda^2$ change. We could solve the model in a general form allowing for serial correlation in residuals and a random system lag and then substitute specific terms for means and covariances using (2), (2a), and (2b). It is much simpler, however, to solve the model (1)-(3) directly.

We first solve for $\tilde{P}_2$ by maximizing $-\gamma E_1(X_2-\hat{X}_2)^2$ with respect to $P_2$ subject to

$$X_2 = \bar{\theta}_1X_1 + \theta_2(2)P_2 + \lambda P_1 + \theta_3(2).$$

$\tilde{P}_2$ must then minimize the sum $\sigma_{X_2}^2 + (\bar{X}_2-\hat{X}_2)^2$. From (2a) it follows

$$\bar{X}_2 = \bar{\theta}_1X_1 + \bar{\theta}_2P_2 + \bar{\lambda}P_1 + \bar{\theta}_3,$$

$$X_2 - \bar{X}_2 = (\theta_2(2)-\bar{\theta}_2)P_2 + (\lambda-\bar{\lambda})P_1 + (\theta_3(2)-\bar{\theta}_3),$$

and

$$\sigma^2_{X_2} = \sigma^2_2 P^2_2 + \sigma^2_\lambda P^2_1 + \sigma^2_3.$$

Minimizing $\sigma^2_{X_2} + (\bar{X}_2 - \hat{X}_2)^2$ with respect to $P_2$ then yields:

$$\tilde{P}_2 = \left(\frac{\bar{\theta}_2}{\sigma^2_2 + \bar{\theta}^2_2}\right)(\hat{X}_2 - \bar{\theta}_1 X_1 - \bar{\lambda} P_1 - \bar{\theta}_3)$$

and

$$\tilde{U}_2(I_1) \equiv -\gamma E_1[X_2(\tilde{P}_2) - \hat{X}_2]^2 = -\gamma[\left(\frac{\sigma^2_2}{\sigma^2_2 + \bar{\theta}^2_2}\right)(\hat{X}_2 - \bar{\theta}_1 X_1 - \bar{\lambda} P_1 - \bar{\theta}_3)^2 + \sigma^2_\lambda P^2_1 + \sigma^2_3].$$

We can now write

$$\begin{aligned} E_0 U &= -E_0(X_1 - \hat{X}_1)^2 + E_0 \tilde{U}_2(I_1) \\ &= -\sigma^2_{X_1} - (\bar{X}_1 - \hat{X}_1)^2 + E_0 \tilde{U}_2(I_1) \\ &= -\sigma^2_2 P^2_1 - \sigma^2_3 - (\bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \bar{\theta}_3 - \hat{X}_1)^2 \\ &\quad -\gamma[(1-H)E_0(\hat{X}_2 - \bar{\theta}_1 X_1 - \bar{\lambda} P_1 - \bar{\theta}_3)^2 + \sigma^2_\lambda P^2_1 + \sigma^2_3] \end{aligned}$$

where the expression for $E_0(X_1 - \hat{X}_1)^2$ is derived as in IIA and

$$H \equiv \frac{\bar{\theta}^2_2}{\sigma^2_2 + \bar{\theta}^2_2}.$$

Substituting for $X_1$ from (2), taking expected values in the expression for $E_0 \tilde{U}_2(I_1)$, and then maximizing the resulting expression for $E_0 U$ with respect to $P_1$ yields:

$$\tilde{P}_1 = \frac{\bar{\theta}_2(\hat{X}_1 - \bar{\theta}_1 X_0 - \bar{\theta}_3) + \gamma(1-H)(\bar{\theta}_1 \bar{\theta}_2 + \bar{\lambda})(\hat{X}_2 - \bar{\theta}^2_1 X_0 - \bar{\theta}_3 - \bar{\theta}_1 \bar{\theta}_3)}{\sigma^2_2 + \bar{\theta}^2_2 + \gamma(1-H)[\bar{\theta}^2_1(\sigma^2_2 + \bar{\theta}^2_2) + 2\bar{\lambda}\bar{\theta}_1 \bar{\theta}_2 + \bar{\lambda}^2] + \gamma\sigma^2_\lambda}.$$

We can write the rule for $\tilde{P}_1$ as

$$\tilde{P}_1 = AG_1 + BG_2$$

where

$$G_1 \equiv (\hat{X}_1 - \bar{\theta}_1 X_0 - \bar{\theta}_3)$$

is the first-period gap between the desired value of goal variable and its expected value with $P_1 = 0$,

$$G_2 \equiv (\hat{X}_2 - \bar{\theta}_1^2 X_0 - \bar{\theta}_3 - \bar{\theta}_1 \bar{\theta}_3)$$

is the second-period gap between the desired value of the goal variable and its expected value with $P_1 = P_2 = 0$,

$$A \equiv \frac{\bar{\theta}_2}{\text{Den}}$$

is the coefficient of policy response to the gap in $X_1$,

$$B \equiv \frac{\gamma(1-H)(\bar{\theta}_1 \bar{\theta}_2 + \bar{\lambda})}{\text{Den}}$$

is the coefficient of policy response to the gap in $X_2$, and

$$\text{Den} \equiv \sigma_2^2 + \bar{\theta}_2^2 + \gamma(1-H)[\bar{\theta}_1^2(\sigma_2^2 + \bar{\theta}_2^2) + 2\bar{\lambda}\bar{\theta}_1\bar{\theta}_2 + \bar{\lambda}^2] + \gamma\sigma_\lambda^2.$$

We observe first that A and B in general are not zero if $\sigma_\lambda^2 < \infty$. That is, as long as the variance of the policy lag is finite, a feedback rule dominates a nonfeedback rule.[21/]

We can now examine how the lag parameters $\bar{\lambda}$ and $\sigma_\lambda^2$ affect the policy response coefficients A and B. Note that Den can be written

$$\text{Den} \equiv \sigma_2^2 + \bar{\theta}_2^2 + \gamma(1-H)[\bar{\theta}_1^2\sigma_2^2 + (\bar{\theta}_1\bar{\theta}_2 + \bar{\lambda})^2] + \gamma\sigma_\lambda^2$$

so that clearly Den > 0. It follows immediately that

$$\frac{\partial|A|}{\partial\sigma_\lambda^2} < 0 \text{ and } \frac{\partial|B|}{\partial\sigma_\lambda^2} < 0.$$

This indicates that policy should respond less vigorously to perceived gaps in goal variables as the variance of the policy lag increases.[22/]

We next compute how A and B change as $\bar{\lambda}$ changes. We find first that

$$\frac{\partial A}{\partial\bar{\lambda}} = \frac{-2\gamma\bar{\theta}_2(1-H)(\bar{\theta}_1\bar{\theta}_2+\bar{\lambda})}{\text{Den}^2}.$$

It follows then that

$$\text{sgn}\left(\frac{\partial|A|}{\partial\bar{\lambda}}\right) = -\text{sgn}(\bar{\theta}_1\bar{\theta}_2+\bar{\lambda}).$$

The last term in parentheses is simply the effect of $P_1$ on the expected value of $X_2$:

$$E_0X_2 = \bar{\theta}_1^2X_0 + \bar{\theta}_2P_2 + (\bar{\theta}_1\bar{\theta}_2+\bar{\lambda})P_1 + \bar{\theta}_1\bar{\theta}_3 + \bar{\theta}_3$$

so that

$$\frac{\partial E_0X_2}{\partial P_1} = \bar{\theta}_1\bar{\theta}_2 + \bar{\lambda}.$$

Our result is then that

$$\frac{\partial|A|}{\partial\bar{\lambda}}\left\{\begin{matrix}>0\\=0\end{matrix}\right\} \Longleftrightarrow \frac{\partial E_0X_2}{\partial P_1}\left\{\begin{matrix}<0\\=0\end{matrix}\right\}.$$

We next find

$$\frac{\partial B}{\partial\bar{\lambda}} = \frac{\gamma(1-H)(\sigma_2^2+\bar{\theta}_2^2+\gamma\sigma_\lambda^2)+\gamma^2(1-H)^2[\bar{\theta}_1^2\sigma_2^2-(\bar{\theta}_1\bar{\theta}_2+\bar{\lambda})^2]}{\text{Den}^2}.$$

Depending on parameter values, $\frac{\partial|B|}{\partial\bar{\lambda}}$ can be positive or negative, but the economic interpretation is not as apparent as for $\frac{\partial|A|}{\partial\lambda}$.

Our results would seem to generalize to multiperiod models as follows:

1. A given change in the shape of the policy lag (represented in our model as a change in $\bar{\lambda}$) can either increase or decrease the responsiveness of optimal policy to new information. The outcome depends on values of parameters of the economic structure as well as on values of parameters of the objective function.
2. An increase in the variability of the policy lag unambiguously reduces the responsiveness of optimal policy to new information. Unless the variance of the policy lag is infinite, however, some responsiveness is better than none.

C. Estimation Uncertainty About the Effects of Policy on Goal Variables

An important consideration in dynamic decision making under uncertainty is the opportunity to learn by doing, that is, to experiment.[23/] In some situations it is possible for the policymaker to learn more about the economic structure by taking an extreme action and then observing the outcome for the goal variable. This opportunity is present, for example, when uncertainty about coefficients is due to estimation. In these situations it may be beneficial to experiment early in the policy horizon, thereby sacrificing on near-term policy goals, in order to gain knowledge about the economic structure, thereby improving the ability to attain policy goals in the future. In this section we will analyze a simple learning-by-doing model and compare the optimal policy which obtains to that which obtains in the inherent uncertainty model.

Before turning to the learning-by-doing model, it is useful to contrast the inherent uncertainty and estimation uncertainty assumptions. Under either assumption the policymaker is assumed to know the distribution of the $\theta$ process. In two-period models this means the policymaker initially knows $E_0\theta(t)$, $t=1, 2$ and $E_0\theta(t)'\theta(t')$, $t,t'=1, 2$. With inherent uncertainty it is assumed that

information from the first period $\langle X_1, P_1 \rangle$ does not alter the distribution of second-period coefficients $\theta(2)$:

$$E_0\theta(2) = E_1\theta(2)$$

and

$$E_0\theta(2)'\theta(2) = E_1\theta(2)'\theta(2).$$

In Bayesian terms this says that the posterior distribution of $\theta(2)$ (the distribution of $\theta(2)$ conditional on $I_1$) is the same as the prior distribution of $\theta(2)$ (the distribution of $\theta(2)$ conditional on $I_0$). With estimation uncertainty these equalities do not hold, so that the prior and posterior distributions of $\theta(2)$ are different. Learning by doing, thus, involves setting $P_1$ to favorably alter the posterior distribution of $\theta_2$.

As an example, suppose that uncertainty about $\theta$ is due to estimation and that the policymaker approaches the first period with ordinary least squares estimators of $\theta(1)$ based on T prior observations of X and P. At the beginning of the second period the OLS estimators of $\theta(2)$ will be based on the prior T observations of X and P and on $X_1$ and $P_1$. Thus, the choice of $P_1$ affects the estimated distribution of $\theta(2)$. By choosing an extreme value of $P_1$, the policymaker can generate an extreme observation which will improve the precision of the estimate of $\theta(2)$.[24/] Thus, there is an incentive to deviate from historical policy, which is not present when there is inherent uncertainty.

Our model of learning by doing captures the essence of this estimation example, but simplifies the problem to allow an explicit solution.[25/] We will assume that $\theta_1$ is known, that $\theta_2$ takes on one of the two values, 1 and 2, with priors p and 1-p, respectively, and that $\theta_3$ is distributed uniformly on [-1,1] and independently over time. We will choose specific targets for the goal variable, $\hat{X}_1 = \bar{\theta}_1 X_0 + 1$ and $\hat{X}_2 = \bar{\theta}_1 \hat{X}_1$, in order to facilitate computation of optimal policy under both types of uncertainty.[26/]

The nature of learning in this model is straightforward. Let $P_1$ be given. If $\theta_2 = 1$, then $X_1$ will be uniformly distributed on the interval $[\bar{\theta}_1 X_0+P_1-1, \bar{\theta}_1 X_0+P_1+1]$. Similarly, if $\theta_2 = 2$, then $X_1$ will be uniformly distributed on the interval $[\bar{\theta}_1 X_0+2P_1-1, \bar{\theta}_1 X_0+2P_1+1]$. Thus, for given $P_1$, $X_1$ will fall in one of three intervals:

$$A = \{y: \bar{\theta}_1 X_0+P_1-1 \leq y < \bar{\theta}_1 X_0+2P_1-1\},$$

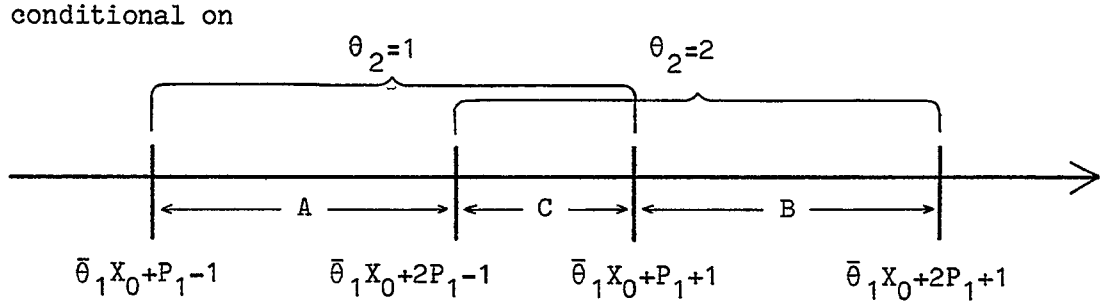
$$B = \{y: \bar{\theta}_1 X_0+P_1+1 < y \leq \bar{\theta}_1 X_0+2P_1+1\}, \text{ and}$$

$$C = \{y: \bar{\theta}_1 X_0+2P_1-1 \leq y \leq \bar{\theta}_1 X_0+P_1+1\},$$

see (Figure 7).

Figure 7

$X_1$ is uniformly distributed on specified interval conditional on




If we observe $X_1 \varepsilon A$, we know that $\theta_2 = 1$, because points in the set A have a zero probability of occurring when $\theta_2 = 2$. Similarly, if we observe $X_1 \varepsilon B$, we know that $\theta_2 = 2$. If we observe, however, $X_1 \varepsilon C$, there is no information upon which to change our priors, because points in C have an equal probability of occurring whether $\theta_1 = 1$ or $\theta_1 = 2$ is true. Thus, if $X_1$ falls in C, we would still attach the probabilities p and 1-p to $\theta_2 = 1$ and $\theta_2 = 2$, respectively.

Learning, in this case, amounts to reducing the probability that $X_1$ falls in C. In order to compute how the probabilities of $X_1$ falling in particular regions depend on $P_1$, we first determine the lengths of the subintervals A, B, and C:

$$\ell(A) = (\bar{\theta}_1 X_0 + 2P_1 - 1) - (\bar{\theta}_1 X_0 + P_1 - 1) = P_1,$$

$$\ell(B) = (\bar{\theta}_1 X_0 + 2P_1 + 1) - (\bar{\theta}_1 X_0 + P_1 + 1) = P_1, \text{ and}$$

$$\ell(C) = (\bar{\theta}_1 X_0 + P_1 + 1) - (\bar{\theta}_1 X_0 + 2P_1 - 1) = 2 - P_1.$$

For $0 \leqq P_1 \leqq 2$, we have

$$\text{prob}(X_1 \varepsilon A | \theta_2 = 1) = 1/2\ell(A) = P_1/2,$$

$$\text{prob}(X_1 \varepsilon C | \theta_2 = 1) = 1/2\ell(C) = 1 - P_1/2,$$

$$\text{prob}(X_1 \varepsilon B | \theta_2 = 2) = 1/2\ell(B) = P_1/2, \text{ and}$$

$$\text{prob}(X_1 \varepsilon C | \theta_2 = 2) = 1/2\ell(C) = 1 - P_1/2.$$

Thus,

$$\text{prob}(X_1 \varepsilon A) = \text{prob}(\theta_2 = 1) \cdot \text{prob}(X_1 \varepsilon A | \theta_2 = 1) = pP_1/2,$$

$$\text{prob}(X_1 \varepsilon B) = \text{prob}(\theta_2 = 2) \cdot \text{prob}(X_1 \varepsilon B | \theta_2 = 2) = (1-p)P_1/2, \text{ and}$$

$$\text{prob}(X_1 \varepsilon C) = \text{prob}(\theta_2 = 1) \cdot \text{prob}(X_1 \varepsilon C | \theta_2 = 1) + \text{prob}(\theta_2 = 2) \cdot \text{prob}(X_1 \varepsilon C | \theta_2 = 2)$$

$$= p(1 - P_1/2) + (1-p)(1 - P_1/2) = 1 - P_1/2.$$

When $P_1 = 0$, $X_1$ will fall in C with probability 1, so that there will be no learning. For $P_1 \geqq 2$, the distributions of $X_1$ conditional on $\theta_2$ have no overlap, so that $X_1$ has a zero probability of falling in C. Since optimal policy [when there is inherent uncertainty] $\tilde{P}_1^I$ is between 0 and 1 (to be shown), optimal policy $\tilde{P}_1$ when there is learning must be between 0 and 2. The lower bound comes from $\tilde{P}_1 \geq \tilde{P}_1^I$. The upper bound comes from the fact that as $\tilde{P}_1$ exceeds $\tilde{P}_1^I$, first-

period loss increases while learning increases. However, there is no learning after $P_1$ reaches 2; that is, $C = \phi$ is the most that can be learned and that occurs at $P_1 = 2$.

Our simple model of estimation uncertainty can be expressed by equations (1) - (3) with

(1) $$\gamma = \frac{V_2}{V_1},\ \hat{X}_1 = \theta_1 X_0 + 1,\ \hat{X}_2 = \theta_1 \hat{X}_1$$

(2a) $$E_0(\theta_1(1),\theta_2(1),\theta_3(1)) = E_0(\theta_1(2),\theta_2(2),\theta_3(2)) = (\bar{\theta}_1, 2-p, 0)$$

$$E_1\theta_1(2) = \bar{\theta}_1,\ E_1\theta_2(2) = \begin{cases} 1 & \text{if } X_1 \varepsilon A \\ 2 & \text{if } X_1 \varepsilon B \\ 2-p & \text{if } X_1 \varepsilon C \end{cases} \qquad E_1\theta_3(2) = 0,$$

where use has been made of--

(i) The mean of a Bernoulli random variable taking on the values $\alpha$ and $\beta$ with probabilities p and 1-p, respectively, is $p \cdot \alpha + (1-p) \cdot \beta$, and for $\theta_2$, $\beta = 2$ and $\alpha = 1$.

(ii) The mean of a random variable which is uniformly distributed on the interval [a,b] is $\frac{1}{b-a}\int_a^b t\,dt = \frac{a+b}{2}$, and for $\theta_3$, b = 1 and a = -1.

(2b) $$E_0(\theta(1)-\bar{\theta}(1))'(\theta(1)-\bar{\theta}(1)) = E_0(\theta(2)-\bar{\theta}(2))'(\theta(2)-\bar{\theta}(2))$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & 1/3 \end{pmatrix},$$

$$E_0(\theta(1)-\bar{\theta}(1))'(\theta(2)-\bar{\theta}(2)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where use has been made of--

(i) The variance of a Bernoulli random variable taking on the values $\alpha$ and $\beta$ with probabilities p and 1-p, respectively, is

$$E(\cdot)^2 - (E(\cdot))^2 = p\alpha^2 + (1-p)\beta^2 - (p\alpha+(1-p)\beta)^2$$

$$= p(1-p)\alpha^2 - 2p(1-p)\alpha\beta + (1-p)(1-(1-p))\beta^2$$

$$= p(1-p)(\alpha-\beta)^2,$$

and for $\theta_2$, $\beta = 2$ and $\alpha = 1$.

(ii) The variance of a random variable which is uniformly distributed on the interval [a,b] is

$$E(\cdot)^2 - (E(\cdot))^2 = \frac{1}{b-a}\int_a^b t^2 dt - (\frac{a+b}{2})^2$$

$$= \frac{1}{b-a}(\frac{b^3-a^3}{3}) - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4(a^2+ab+b^2)}{12} - \frac{3(a^2+2ab+b^2)}{12} = \frac{a^2-2ab+b^2}{12} = \frac{1}{12}\cdot(b-a)^2,$$

and for $\theta_3$, $b = 1$ and $a = -1$.

(iii) $\theta_2$ and $\theta_3$ are independently distributed, implying

$$E_0\theta_2(t)\theta_3(t') - E_0\theta_2(t)E_0\theta_3(t') = 0, \text{ for } t,t'=1, 2.$$

(iv) $\theta_3$ is independently distributed over time, implying

$$E_0\theta_3(1)\theta_3(2) - E_0\theta_3(1)E_0\theta_3(2) = 0.$$

(v) $\theta_2$ is a fixed, but unknown, variate, implying

$$E_0\theta_2(1)\theta_2(2) - E_0\theta_2(1)E_0\theta_2(2) =$$

$$p\alpha^2 + (1-p)\beta^2 - (p\alpha+(1-p)\beta)^2 = p(1-p)(\alpha-\beta)^2,$$

since $\text{prob}[\theta_2(2)=\alpha|\theta_2(1)=\alpha] = 1$ and $\text{prob}[\theta_2(2)=\beta|\theta_2(1)=\beta] = 1$, and for $\theta_2$, $\beta = 2$ and $\alpha = 1$.

We also have

$$E_1(\theta(2)-\bar{\theta}(2))'(\theta(2)-\bar{\theta}(2)) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} & \text{if } X_1 \varepsilon A \cup B \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & 1/3 \end{pmatrix} & \text{if } X_1 \varepsilon C \end{cases}$$

and

(3) $\underline{X}_1 = X.$

Since learning is possible only when there is more than one period, we shall assume $\gamma > 0$ and solve for $\tilde{U}_2(I_1)$. Recall $U_2(P_2,I_1) = -\gamma E_1(X_2-\hat{X}_2)^2$, $\tilde{P}_2$ maximizes $U_2(P_2,I_1)$, and $\tilde{U}_2(I_1) \equiv U_2(\tilde{P}_2,I_1)$. We now compute $\tilde{P}_2$ and $\tilde{U}_2(I_1)$ depending on the outcome for $X_1$.

If $X_1 \varepsilon A$, $\tilde{P}_2$ is found by maximizing with respect to $P_2$:

$$-E_1(X_2-\hat{X}_2)^2$$

subject to $X_2 = \bar{\theta}_1 X_1 + P_2 + \theta_3(2)$. We know from the certainty equivalence model IA (p. 21) that the solution to this problem is

$$\tilde{P}_2 = \hat{X}_2 - \bar{\theta}_1 X_1$$

and

$$\tilde{U}_2(I_1) = -\gamma\sigma_3^2.$$

If $X_1 \varepsilon B$, $\tilde{P}_2$ is found by maximizing with respect to $P_2$:

$$-E_1(X_2-\hat{X}_2)^2$$

subject to $X_2 = \bar{\theta}_1 X_1 + 2P_2 + \theta_3(2)$. Again we know from the certainty equivalence model IA (p. 21) that the solution to this problem is

$$\tilde{P}_2 = \frac{\hat{X}_2-\bar{\theta}_1X_1}{2}$$

and

$$\tilde{U}_2(I_1) = -\gamma\sigma_3^2.$$

Finally, if $X_1\varepsilon C$, $\tilde{P}_2$ is found by maximizing with respect to $P_2$:

$$-E_1(X_2-\hat{X}_2)^2$$

subject to $X_2 = \bar{\theta}_1X_1 + \theta_2(2)P_2 + \theta_3(2)$ where $\overline{\theta_2(2)}$ and $\sigma_2^2$ are given in (2a) and (2b). We know from the inherent uncertainty model IIA (p. 55) that the solution to this problem is

$$\tilde{P}_2 = \left(\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_2-\bar{\theta}_1X_1)$$

and

$$\tilde{U}_2(I_1) = -\gamma\left[\sigma_3^2+\left(\frac{\sigma_2^2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_2-\bar{\theta}_1X_1)^2\right]$$

which in our case is

$$\tilde{P}_2 = \left(\frac{2-p}{4-3p}\right)(\hat{X}_2-\bar{\theta}_1X_1)$$

and

$$\tilde{U}_2(I_1) = -\gamma\left[\sigma_3^2+\left(\frac{p(1-p)}{4-3p}\right)(\hat{X}_2-\bar{\theta}_1X_1)^2\right].$$

Define $N(p) = p(1-p)$, $D(p) = 4-3p$, and $\Psi(p) = \frac{N(p)}{D(p)}$. Since

$$\begin{aligned} E_0\tilde{U}_2(I_1) = {} & \text{prob}(X_1\varepsilon A)E_0[\tilde{U}_2(I_1)|X_1\varepsilon A] + \\ & \text{prob}(X_1\varepsilon B)E_0[\tilde{U}_2(I_1)|X_1\varepsilon B] + \\ & \text{prob}(X_1\varepsilon C)E_0[\tilde{U}_2(I_1)|X_1\varepsilon C], \end{aligned}$$

we have

$$\begin{aligned} E_0\tilde{U}_2(I_1) &= -\left(\frac{pP_1}{2}\right)\gamma\sigma_3^2 - \left(\frac{(1-p)P_1}{2}\right)\gamma\sigma_3^2 - \left(\frac{2-P_1}{2}\right)\gamma[\sigma_3^2+\Psi(p)E_0(\hat{X}_2-\bar{\theta}_1X_1)^2] \\ &= -\gamma\sigma_3^2 - \gamma\Psi(p)\left(1 - \frac{P_1}{2}\right)E_0(\hat{X}_2-\bar{\theta}_1X_1)^2. \end{aligned}$$

Optimal first-period policy is found by maximizing with respect to $P_1$

$$-E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1)$$

$$= -\sigma_2^2P_1^2 - \sigma_3^2 - (\bar{\theta}_1X_0+\bar{\theta}_2P_1-\hat{X}_1)^2 - \gamma\sigma_3^2 - \gamma\Psi(p)(1 - \frac{P_1}{2})E_0(\hat{X}_2-\bar{\theta}_1X_1)^2$$

where use has been made of the inherent uncertainty model IIA (p. 55).

The first-order condition for $\tilde{P}_1$ to be a maximizer is a quadratic expression in $\tilde{P}_1$, which yields the two solutions:

$$\tilde{P}_1 = \frac{2\gamma\bar{\theta}_1^2\Psi(p)(3-2p)+D(p)\pm\sqrt{\gamma^2\Psi(p)^2\bar{\theta}_1^4(8-15p+7p^2)+\gamma\Psi(p)D(p)\bar{\theta}_1^2(6-5p)+D(p)^2}}{(3/2)\gamma\bar{\theta}_1^2N(p)}.$$

We now want to analyze this solution and compare it to the one under inherent uncertainty. First, we can show that for $\gamma\bar{\theta}_1^2 < 18$, the larger of the two roots is greater than 2 and can be ruled out (see pp. 67-68). Note that for the larger root

$$\tilde{P}_1^+ > \frac{2\gamma\bar{\theta}_1^2\Psi(p)(3-2p)+D(p)+\sqrt{[7^{\frac{1}{2}}(1-p)\gamma\bar{\theta}_1^2\Psi(p)+D(p)]^2}}{\frac{3}{2}\gamma\bar{\theta}_1^2N(p)}.$$

$$= \frac{2[(6+7^{\frac{1}{2}})-(4+7^{\frac{1}{2}})p]}{3(4-3p)} + \frac{4(4-3p)}{3\gamma\bar{\theta}_1^2p(1-p)}.$$

The first term is greater than 4/3 for $p < 1$, while the second term is greater than 2/3 for $\gamma\bar{\theta}_1^2 < 18$. Since normally we would expect both $\gamma$ and $\bar{\theta}_1$ to be less than 1, we can restrict our attention to the smaller root.

Second, note that with $\hat{X}_2 = \bar{\theta}_1\hat{X}_1$ the optimal first-period policy when there is inherent uncertainty is given by

$$\tilde{P}_1^I = (\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2})(\hat{X}_1-\bar{\theta}_1X_0) \text{ (see p. 56)}$$

$$= \frac{2-p}{4-3p} \text{ (recalling } \hat{X}_1 = \bar{\theta}_1X_0+1).$$

Third, note that the optimal first-period policy when there is estimation uncertainty can be expressed as

$$\tilde{P}_1 = \frac{2\gamma\bar{\theta}_1^2\Psi(p)(3-2p)+D(p)-\sqrt{[\gamma\bar{\theta}_1^2\Psi(p)(3-\frac{5p}{2})+D(p)]^2+\gamma^2\bar{\theta}_1^4\Psi(p)^2(\frac{3p^2}{4}-1)}}{(3/2)\gamma\theta_1^2 p(1-p)}.$$

Finally, it is straightforward to show that if we arbitrarily set $\gamma^2\bar{\theta}_1^4\Psi(p)^2(\frac{3p^2}{4}-1)$ to zero, the expression for $\tilde{P}_1$ simplifies to $\tilde{P}_1 = \frac{2-p}{4-3p}$, which is the inherent uncertainty solution. Thus, returning to the actual expression for $\tilde{P}_1$, we see that the term under the radical sign,

$$\gamma^2\bar{\theta}_1^4[\frac{p(1-p)}{4-3p}]^2[\frac{3p^2}{4}-1],$$

represents the adjustment for learning to the inherent uncertainty solution. Since $\frac{3p^2}{4}-1<0$, the rest of the terms are squared and thus positive, and the radical is preceded by a negative sign, it follows that the adjustment for learning adds a positive amount to the inherent uncertainty solution, $\tilde{P}_1 > \tilde{P}_1^I$.

By inspection it is clear that the adjustment for learning increases when $\gamma$ or $\bar{\theta}_1$ increases. When $\gamma$ increases, the future is more important to the policymaker, that is, $E_0\tilde{U}_2(I_1)$ has a greater weight in the policymaker's objective function. Thus, learning is more important, since it allows a higher value of $E_0\tilde{U}_2(I_1)$ to be attained.

When $\bar{\theta}_1$ increases, the expected value of $\tilde{P}_2$ conditional on $P_1 = 0$ increases whether $X_1 \varepsilon A$, B, or C:

$$E_0(\tilde{P}_2|P_1=0)\begin{cases} \tilde{X}_2-\bar{\theta}_1^2X_0 = \bar{\theta}_1 & X_1\varepsilon A \\ \dfrac{\tilde{X}_2-\bar{\theta}_1^2X_0}{2} = \dfrac{\bar{\theta}_1}{2} & X_1\varepsilon B \\ (\frac{2-p}{4-3p})(\hat{X}_2-\bar{\theta}_1^2X_0) = (\frac{2-p}{4-3p})\bar{\theta}_1 & X_1\varepsilon C. \end{cases}$$

But a larger value of $\tilde{P}_2$ implies greater variance of the goal variable $X_2$ if $X_1 \varepsilon C$. This variance can be avoided if $X_1$--that is, if there is learning. To summarize, a greater value of $\bar{\theta}_1$ implies a greater expected value of $\tilde{P}_2$, which implies more variance in $X_2$, unless there is certainty about $\theta_2(2)$. Thus, an increase in $\bar{\theta}_1$ increases the value of learning.

We would expect also that the greater the initial uncertainty about $\theta_2$, the greater would be the value of learning. But given our assumed probability distribution of $\theta_2$, it is not possible to increase the variance of $\theta_2$, p(1-p), without also changing the mean of $\theta_2$, 2-p. Thus, we can not determine how $\tilde{P}_1$ changes when $\sigma_2^2$ changes because $\bar{\theta}_2$ also changes. Since p(1-p) does appear with a positive sign in the adjustment for learning term, though, it at least suggests that the expected result would obtain with more general probability distributions of $\theta_2$.

Finally, the question arises whether learning by doing can overturn our earlier policy implication that uncertainty requires caution. Earlier we found that optimal policy should respond less to perceived gaps in goal variables from their targets the greater is the uncertainty about policy. Optimal policy with inherent uncertainty must always be less, in absolute value, than the optimal policy when the policy coefficient is known, the certainty equivalence policy. So can the optimal policy under estimation uncertainty ever be greater, in absolute value, than the optimal certainty equivalence policy? The anwser is yes.

The optimal first-period policy when $\theta_2$ is known is

$$\tilde{P}_1^K = \frac{1}{\theta_2}(\hat{X}_1 - \bar{\theta}_1 X_0) = \frac{1}{2-p}(\hat{X}_1 - \bar{\theta}_1 X_0).$$

If we stick to the assumption that $\hat{X}_1 - \theta_1 X_0 = 1$, it appears that the optimal learning policy $\tilde{P}_1$ can never exceed $\tilde{P}_1^K$. Although $\tilde{P}_1$ increases with $\gamma$, the limit of $\tilde{P}_1$ as $\gamma \to \infty$ is $\tilde{P}_1^K$.

If, however, we set $\hat{X}_1 = \theta_1 X_0$, then both the certainty equivalence solution $\tilde{P}_1^K$ and the inherent uncertainty solution $\tilde{P}_1^I$ are zero. The optimal solution under estimation uncertainty $\tilde{P}_1$ in this case is positive, because the adjustment for learning to the inherent uncertainty solution is still positive. Since all solutions change continuously with the gap $\hat{X}_1 - \bar{\theta}_1 X_0$, there is a small gap $\hat{X}_1 - \bar{\theta}_1 X_0 > 0$ for which $\tilde{P}_1^I < \tilde{P}_1^K < \tilde{P}_1$. Learning, then, can overturn the implication that uncertainty requires caution.

Footnotes

1/ The survey is not intended to be exhaustive; it includes only a sample of prominent papers on a few key issues.

2/ See, for example, Wallace (1976) and Lucas (1978).

3/ The solution can also be expressed as a feedback rule which states how policy variables should be set each period based on available information.

4/ In the case of $n > 1$ goal variables, the terms $V_i$ and $(X_i-\hat{X}_i)^2$ are 1xn and nx1 matrices, respectively. The quadratic objective function assumed here is not general in that it does not allow cross-product terms either over time or between variables at a point in time.

5/ The variable $\gamma$ is $\frac{1}{1+r}$, where $r \geqq 0$ is the policymakers' rate of time preference.

6/ In the information lags and multiple goal variable models $X_t$ can be thought of as an n-dimensional vector, and the process becomes

$$X_{it} = \sum_{j=1}^{n} \theta_{ij}(t)X_{jt-1} + \theta_{i,n+1}(t)\cdot P_t + \theta_{i,n+2}(t) \quad i=1,\ldots,n.$$

In more compact notation the process can be written as before:

$$X_t = \theta_1(t)X_{t-1} + \theta_2(t)P_t + \theta_3(t)$$

for all t, where now

$$X_t = \begin{pmatrix} X_{1t} \\ \cdot \\ \cdot \\ \cdot \\ X_{nt} \end{pmatrix}, \quad \theta_1(t) = \begin{pmatrix} \theta_{11}(t) & \cdots & \theta_{1n}(t) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \theta_{n1}(t) & \cdots & \theta_{nn}(t) \end{pmatrix}, \quad \theta_2(t) = \begin{pmatrix} \theta_{1,n+1}(t) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n,n+1}(t) \end{pmatrix},$$

$$\theta_3(t) = \begin{pmatrix} \theta_{1,n+2}(t) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n,n+2}(t) \end{pmatrix}$$

7/ Suppose, however, we search initially for the optimal first-period rule $\tilde{f}_1$ over all functions mapping an arbitrary $I_0$ into $P_1$. Applying $\tilde{f}_1$ to the known vector $I_0$ yields optimal first-period policy $\tilde{P}_1$ as a scalar, $\tilde{P}_1 = \tilde{f}_1(I_0)$.

8/ Footnote 7 applies in this case to $g_1$ and $g_2$.

9/ Let N denote the vector of realizations of economic variables observed at t=1 but not at t=0 (i.e., N=new information), so that for each $I_0$ and $I_1$, N is defined by: $N = I_1 \sim I_0$. Thus, $I_1 = I_0 \cup N$. For arbitrary $\langle g_1, g_2 \rangle \varepsilon G$ we can extend $g_2$ to $\{I_1\}$ by $\bar{g}_2(I_1) \equiv \bar{g}_2(I_0 \cup N) = g_2(I_0)$ for all N. Then it is clear that each pair of functions $\langle g_1, \bar{g}_2 \rangle$ contained in $G$ is also contained in $F$. However, as long as the economic process is not deterministic and there are new observations at t=1, there exists a pair $\langle f_1, f_2 \rangle \varepsilon F$ such that $f_2(I_0 \cup N) \neq f_2(I_0 \cup N')$ for $N \neq N'$. The pair $\langle f_1, f_2 \rangle$ is contained in $F$ but not in $G$.

10/ For one-period problems $V_2$ is set equal to zero and $E_0U$ is maximized with respect to $P_1$. X can be considered a vector with n components as in footnote 6 or a scalar (n=1). When n > 1, however, $\sum$ must be defined differently. As in footnote 6,

$$\theta(t) = (\theta_1(t), \theta_2(t), \theta_3(t)) = \begin{pmatrix} \theta_{11}(t) & \cdot & \cdot & \theta_{1n}(t) & \theta_{1,n+1}(t) & \theta_{1,n+2}(t) \\ \vdots & & & \vdots & \vdots & \vdots \\ \theta_{n1}(t) & \cdot & \cdot & \theta_{nn}(t) & \theta_{n,n+1}(t) & \theta_{n,n+2}(t) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \theta_{1R}(t) \\ \\ \theta_{nR}(t) \end{pmatrix},$$

where $\theta_{iR}$ is the $i^{th}$ row vector of $\theta$. The variance-covariance matrix of $\theta$ is defined by

$$\sum(t,t')_s = E_s \begin{pmatrix} (\theta_{1R}(t)-E_s\theta_{1R}(t))' \\ \\ (\theta_{nR}(t)-E_s\theta_{nR}(t))' \end{pmatrix}_{n(n+2)x1} \underset{1xn(n+2)}{(\theta_{1R}(t')-E_s\theta_{1R}(t'),\ldots,\theta_{nR}(t')-E_s\theta_{nR}(t'))}$$

$$\begin{pmatrix} \sum_{11}(t,t')_s & \cdots & \sum_{1n}(t,t')_s \\ \vdots & & \vdots \\ \sum_{n1}(t,t')_s & \cdots & \sum_{nn}(t,t')_s \end{pmatrix},$$

where $\sum_{ij}(t,t')_s$ is the covariance of $t^{th}$ period, $i^{th}$ row coefficients with $t'^{th}$ period $j^{th}$ row coefficients based on information at time s.

11/ Theil (1965), pp. 404-424.

12/ Theil (1965), pp. 508-510. Our two-period model with one goal variable and one policy control variable can be written in the form

$$\begin{pmatrix} X_1 \\ \\ X_2 \end{pmatrix} = \begin{pmatrix} \theta_2(1) & 0 \\ \\ \theta_1(2)\theta_2(1) & \theta_2(2) \end{pmatrix} \begin{pmatrix} P_1 \\ \\ P_2 \end{pmatrix} + \begin{pmatrix} \theta_1(1)X_0+\theta_3(1) \\ \\ \theta_1(2)\theta_1(1)X_0+\theta_1(2)\theta_3(1)+\theta_3(2) \end{pmatrix}$$

$$\equiv \begin{pmatrix} R_{11} & 0 \\ \\ R_{21} & R_{22} \end{pmatrix} P + \mu.$$

13/ It has implicitly been assumed that $\bar{\theta}_{23} > 0$, but this is not an important restriction. If both $\bar{\theta}_{13}$ and $\bar{\theta}_{23}$ were zero, there would be no effect of policy on either goal variable and one choice of policy would be as good as any other. So we want to assume that at least one of these coefficients is not zero. Since the model is symmetric with respect to X and $\pi$, it could just as easily have been assumed that $\bar{\theta}_{13} > 0$.

14/ The economic process may not be stable under both choices, however. In rational expectations-natural rate models attempts to fix a nominal interest rate make the price level indeterminate.

15/ This is essentially the model found in Poole (1970).

16/ For a more complete exposition of these points see Kareken-Miller (1976).

17/ The two-period model described here is a special case of the general model found in Kareken-Muench-Wallace (1973).

18/ This model can be found in Brainard (1967).

19/ Friedman seemed to make this type of argument in his 1953 article.

20/ For the bulk of the paper Fischer and Cooper assume that the lag weights can be expressed as:

$$w_i(t) = (1-\beta)(1-\lambda_{t-i}) \prod_{j=0}^{i-1} \lambda_{t-j}$$

where $\prod_{j=0}^{-1} (\cdot) \equiv 1$ and for all s

$$\lambda_s = \lambda + \varepsilon_s, \; 0 < \lambda < 1$$

$$E\varepsilon_s = 0, \; E\varepsilon_s^2 = \sigma_\varepsilon^2.$$

Under known lags $\sigma_\varepsilon^2 = 0$ and the w's are from a Koyck lag distribution

$$w_i = (1-\beta)(1-\lambda)\lambda^i.$$

The length of lag is defined by the mean lag

$$\frac{\sum_{i=0}^{\infty} w_i(t) \cdot i}{\sum_{i=0}^{\infty} w_i(t)},$$

and the variance of the lag is defined in terms of the parameter $\sigma_\varepsilon^2$.

The mean lag increases in the known lag case as $\lambda$ increases. This implies that $w_0$, the known coefficient on current policy, decreases as the length of lag increases. The model in the known lag case is simply a version of our known coefficients model in IA with

$$\theta_1 = \beta,$$

$$\theta_2 = (1-\beta)(1-\lambda), \text{ and}$$

$$\theta_3(t) = (1-\beta)(1-\lambda)\sum_{i=1}^{\infty}\lambda^i P_{t-i} + U(t).$$

Optimal policy in model IA is given by

$$\tilde{P}_t = \frac{1}{\theta_2}(\hat{X}_t - \theta_1 X_{t-1} - \bar{\theta}_3(t))$$

$$\equiv \frac{1}{(1-\beta)(1-\lambda)}(\hat{X}_t - \beta X_{t-1} - (1-\beta)(1-\lambda)\sum_{i=1}^{\infty}\lambda^i P_{t-i}).$$

The deviation of $\tilde{P}$ from zero needed to close the gap grows as $\theta_2$ declines. Since $\theta_2$ declines with an increase in the mean lag, it is not surprising then that Fischer and Cooper find the longer the lag the more vigorously stabilization policy should be used.

21/ New information is represented by the variable $X_0$. The response of $\tilde{P}_1$ to new information is then $-\bar{\theta}_1 A - \bar{\theta}_1^2 B$. As long as not both A and B are zero, which will be the case in general with $\sigma_\lambda^2 < \infty$, a feedback rule adjusting $P_1$ to new information is seen to dominate a nonfeedback rule.

22/ This proposition could be restated in terms of the previous footnote to say that policy should respond less vigorously to new information as the variance of the policy lag increases.

23/ For "learning by doing" models, see Chow [1975], Prescott, and Zellner.

24/ The result that extreme observations improve the precision of OLS estimates can be found in Johnston.

25/ The estimation problem referred to in the example above cannot be solved explicitly for optimal policy--even in the two-period case.

26/ By assuming a positive gap between $\hat{X}_1$ and $\bar{\theta}_1 X_0$, we have assured that the optimal policy $\tilde{P}_1^I$ when $\theta_2$ is inherently uncertain is less than the optimal policy $\tilde{P}_1^K$ when $\theta_2$ is known (see page 51).

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