The Capital Stock Modified Competitive Equilibrium

Martin Shubik*

1. Introduction

The general equilibrium model is static. Even when defined for T time periods the true flavor of the dynamics of an ongoing economy is missed. At the end of the T^{th} period, any stock left over is of zero worth.

The model presented here contains the usual general equilibrium model as a special case of an economy which uses money and which has an interest rate of $\rho = 0$. When $\rho > 0$, capital stock is left over with positive worth to the future generations.

Even the model suggested here, which follows the dynamic programming idea of a salvage value for leftover stock, fails to capture some of the basic aspects of a multigenerational economy. We would like to consider the implications of an overlap such as is shown in Figure 1a where several generations whose lives overlap have to take care of the young and the old. Setting aside the extra problems posed by such a model, we may regard our treatment as covering the case with successive but not overlapping generations. When one generation dies, the next is born and begins life with the capital stock left behind.

There are several different ways we can explain the transfer of capital stock to successive generations. We may consider interlinked utility functions, love or altruism; the external imposition of law; or the design of a self-policing noncooperative game. The example presented in this paper is closest to the latter two approaches and is a direct extension of the type of models constructed previously by Shubik (1973), Shapley and Shubik (1977), and Dubey and Shubik (1977a,b).[1]

2. The Model

2.1. *Competitive Equilibrium*

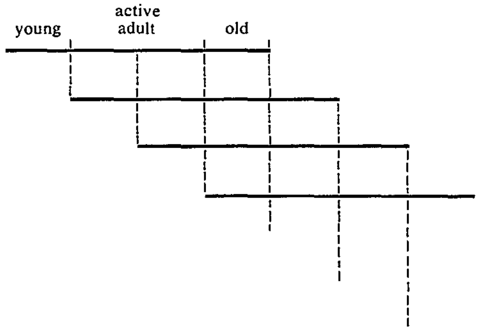
Since in what follows there is a considerable amount of notation and tedious

*This work relates to U.S. Department of the Navy Contract N00014-77-C-0518 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the U.S. Government, and no official endorsement should be inferred.

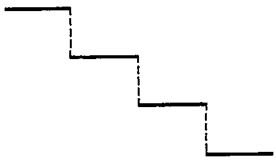
[1]Author names and years refer to the works listed at the end of this book.

Figure 1

1a



1b



detail concerning accounting conventions, we present a specific example in full detail.

Consider a society with a continuum of small traders who live for three periods. All traders are identical, and all firms are identical. Each trader i has a utility function of the form

$$(1)\qquad U^i = \sum_{t=1}^{T} \beta^{t-1}\phi(c_t^i) \text{ specified to } \sum_{t=1}^{3} \beta^{t-1}\sqrt{c_t^i}$$

where

c_t^i = amount consumed by i at time t
β = a natural discount factor.

At $t = 1$ the economy is organized, a central bank comes into being, and private banks and corporations are set up.

Because our prime purpose is the study of the rate of interest for money, the quantity of money, and the flows of money, we make some heroic simplifications. All simplifications have been considered and for the main purpose at hand can be defended.

- There is no exogenous uncertainty.
- We consider a single-commodity world.
- The managers of the firms are automata who maximize long-run (finite-horizon) profits.
- All traders are alike and small.
- All firms are alike and small.
- All inside banks are alike and small.
- We consider only relatively simple strategies.[2]
- We consider only finite-period economies.

We are primarily concerned with comparing our monetary economies at equilibrium with a related general equilibrium model, so we first solve for the

[2]The reader not interested in the delicate aspects of historical strategies in multistage games should ignore this comment. However, it is noted as a warning that equilibria other than those noted may exist.

latter. Because of the extreme symmetry imposed on our model, we may drop identifying superscripts on variables and parameters of the firms, individuals, and banks.

Let us assume that the typical firm has a production function of the form

(2) $k_t = g(I_{t-1})$ specified to $k_t = \sqrt{\alpha I_{t-1}}$

where

I_t = the amount invested in time t
k_t = the amount of capital at the start of time t.

We have

(3) $k_t = I_t + c_t.$

Let A = the initial supply of capital at time t; hence $k_1 = A$. At the start of any period, capital is eaten or used to produce more capital at the start of the next period (inventorying is ruled out).

Each individual has shares in the firms. Individual i holds the shares of firm j with a density of $f^i_{j,t}$ at time t. Since all firms and individuals are the same, in equilibrium there will be no trade in shares, and we can denote the typical portfolio of an individual by

f = the portfolio of an individual in stock of the firms.

The competitive equilibrium is given by prices p_1 (= 1), p_2, p_3 satisfying

(4) $\max \sqrt{c_1} + \beta\sqrt{c_2} + \beta^2\sqrt{c_3}$

subject to

$$c_1 + p_2 c_2 + p_3 c_3 = f\Pi$$

and

(5) $\max \Pi = A - I_1 + p_2\left(\sqrt{\alpha I_1} - I_2\right) + p_3\sqrt{\alpha I_2}$

where

(6) $c_1 = A - I_1$

(7) $c_2 = \sqrt{\alpha I_1} - I_2$

(8) $c_3 = \sqrt{\alpha I_2}.$

The equilibrium prices are p_1, p_2, p_3. We may set $p_1 = 1$. The total profit for any firm is Π. In (4) it refers to total industry profit. Thus $f\Pi$ is the income of an individual. The marginal utility of income (the Lagrangian multiplier) is μ.

We have these six conditions on the maximization of welfare and profits:

$$(9) \qquad \frac{1}{2\sqrt{c_1}} = \mu$$

$$(10) \qquad \frac{\beta}{2\sqrt{c_2}} = \mu p_2$$

$$(11) \qquad \frac{\beta^2}{2\sqrt{c_3}} = \mu p_3$$

$$(12) \qquad c_1 + p_2 c_2 + p_3 c_3 = f\Pi$$

$$(13) \qquad \frac{p_2 \alpha}{2\sqrt{\alpha I_1}} = 1$$

$$(14) \qquad \frac{p_3 \alpha}{2\sqrt{\alpha I_2}} = p_2.$$

Fixing α=400, we calculate the cases A = 100, 200, 400 with β = 1 and A = 100 with β = 1/2 (see Table 1).

Table 1

A	β	c_1	c_2	c_3	p_1	p_2	p_3	μ	k_1	k_2	k_3
100	1	46.81104	88.00820	152.1228	1	.729311	.5547206	.073080	100	145.8616	152.1228
200	1	108.96850	119.77190	168.5800	1	.953834	.8039840	.047898	100	190.8200	168.5800
400	1	246.76040	161.03033	186.0620	1	1.237890	1.1516200	.031830	100	247.5780	186.0620
100	½	60.32134	100.84180	100.2800	1	.386710	.5013990	.064378	100	125.9820	100.2800

2.2. *The Noncooperative Game: Modeling Considerations*

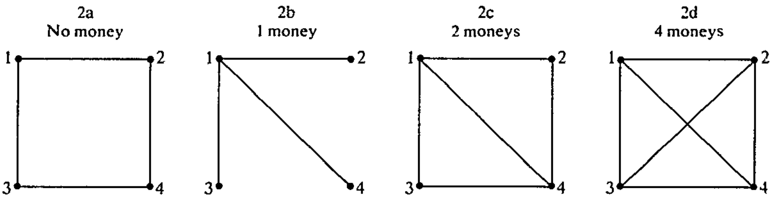
We may attempt to model the same economic problem as a game of strategy. In order to do so we must specify the structure of trade. A natural consideration in defining a game of strategy is to introduce a money.

A *money* is defined as a good or item which can be traded directly for all other items which are traded. Given m commodities to be traded, depending on the number of markets which exist, there may be 0, 1, 2, ..., $m-2$, or m moneys. For m = 4, Figure 2 shows cases with 0, 1, 2, and 4 moneys. If we allow for the existence of all pairwise markets, then there will be $m(m-1)/2$ markets and m moneys. (In Figure 2 a point is a good and a line joining two points symbolizes a market between the two goods at either end.)

If all items are moneys, then there is no need for credit because all the wealth of all individuals can be used in direct exchange. This is not so if there are few markets. A difference between the total wealth of an individual and that person's cash or immediate money position appears.

There are many reasons (involving such things as search, transportation, and trust) why considerably fewer than all markets exist in any economy. Without further discussion, we assume that in the economies we examine if

Figure 2



there are m commodities then there will be m markets and one money.

If the money is also a commodity, its value in consumption or production serves as its backing. In that case, we are examining an economy with $m+1$ commodities, one of which serves as money. (Figure 2b shows such a structure, with the first commodity acting as a money.) If the money is fiat, or some form of credit with no direct physical properties of a commodity, then rules must be supplied as to how it enters the economy and how it is redeemed at the end of trade. This will call for a specification of rules concerning failure to repay debts.

In essence, the model constructed below is based on the following factors, which are explained in more detail subsequently:

1. A satisfactory model of a competitive economy must have a mechanism for price formation which depends on the strategies of the agents.
2. A modern economy has at least one money, that is, one commodity or financial instrument which can be exchanged directly for any other economic good.
3. In an economy with trade in m goods with less than $m(m-1)/2$ markets, a necessary and sufficient condition for the competitive allocations to be feasible given only one round of trade is that at least one commodity is a money which is in sufficient supply.
4. A sufficient supply of a money can always be guaranteed by the societal specification of rules of issue and rules concerning penalties for failure to repay.
5. The use of a money and credit system provides a strategic decoupling which enables individuals to move independently simultaneously in equilibrium or disequilibrium. Thus the aggregate economic process may appear to take place at the same time whereas individual behavior is sequential.
6. Even if we model time as continuous, the individual sequential process of trade in money creates a transactions holding or float which is positive over the finite period of time during which trade takes place.
7. For an economy which exists for a finite number of time periods, a necessary condition for individuals to be able to achieve a competitive outcome is that the money rate of interest $\rho = 0$. If $\rho > 0$, this implies that individuals are taxed for using money. If the original issue of money is from the government or outside bank, this implies that there will be a positive drain from the system, and the usual income-equals-expenditure condition of the competitive allocation cannot hold.

8. If the penalties for failure to repay loans are set appropriately, individuals will be motivated to avoid ending in debt.
9. Assume that an economy lasts T time periods. We introduce the fiction of a $T+1$ period in which the referee punishes those who end in debt and also offers to buy at given prices any inventories which are left over. Then for any set of positive prices p_{T+1} set for the purchase of leftover capital, there will be associated a positive rate of interest for money which balances the residual debt of the economy at time $T+1$ against the value of the remaining capital stock.
10. If the prices p_{T+1} are set to zero, then the noncooperative equilibria of the nonatomic game include the competitive equilibria, there is no value to capital stock left over, and the rate of interest $\rho = 0$.
11. The money rate of interest needed to call forth any residual capital stock is also a function of the trading technology.
12. In a multistage economy without trust or other credit and without uncertainty, money serves at least two purposes. It serves to finance transactions and to finance intertemporal trade. In a single-stage economy, money is needed only to finance transactions. A simple exchange of personal IOU notes for government notes serves to provide money to finance transactions. This, however, will not cover the financing called for by intertemporal trade. In particular, different quantities of money may be required at different periods.
13. We may consider a system with two types of money which exchange on a one-for-one basis but differ in their way of entry into and exit from the system. In particular, we may create an internal or privately held banking system by having bank shares issued and sold after the issue of outside money. This may provide for a flexible system to finance intertemporal trade and, if the rate of interest is positive, result in the profits of the banking system being paid back to the traders.

2.3. *The Noncooperative Game: Description and Notation*

2.3.1. *Initial Conditions*

At the start of the economy we imagine a set of traders with an initial endowment of capital with density A. No further input comes into the economy, so their ownership can be characterized by $(A,0,0)$—A being the exogenous endowment for the first period and 0, the endowment for the subsequent two periods.

In a slightly more complex model we might wish to consider a vector of initial resources A of m dimensions plus a unit of time. Thus we might imagine both production and consumption as nothing more than a mapping of $(A_t,1) \rightarrow A_{t+1}$ with time being used up and replenished exogenously. A_t is the vector of resources at the start of time t, excluding time. $(A_t,1)$ is the vector of all resources including time.

It is useful to regard all economic entities as existing in pairs for accounting purposes. Thus at the start of this economy the accounts of the individuals appear as in Figure 3.

In general, individuals do not bother to issue themselves with ownership paper for any assets except land, houses, and major durables. Thus frequently the amount of ownership paper in an economy is less than the ownership.

2.3.2. *The Organization of the Economy*

We must describe the formation of and rules for guiding the outside bank, inside banks, and firms. There are many different ways in which they can be formed. It is my belief that a useful way to proceed is to select one way, thereby producing a completely well-defined model, and then to investigate the changes caused by considering variations in the formation of institutions.

Figure 3

Individuals

Assets	Debt, Equity
A	
	A

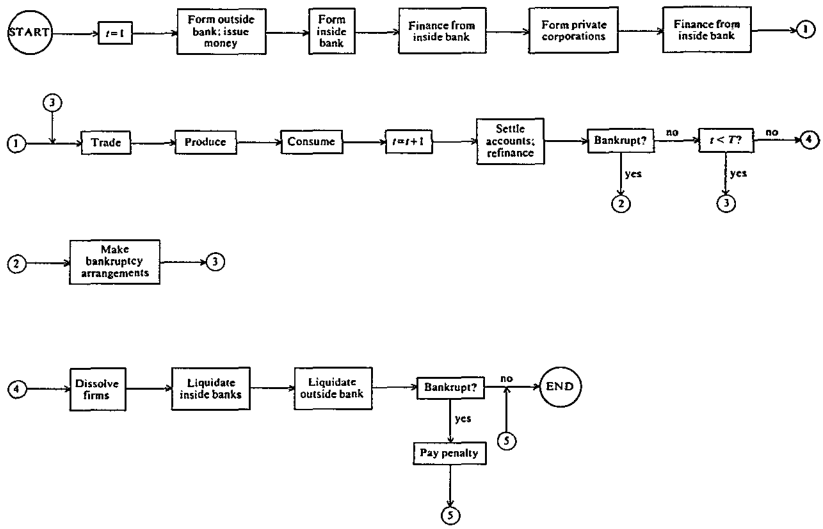
Figure 4 shows one possible pattern in the formation and liquidation of the institutions in an economy. The institutions we need are an outside bank, inside banks, and firms. The financial instruments are outside money, bank money, bank and corporate shares, and personal IOU notes or credit.

2.3.3. *The Outside Bank*

We imagine a society which has decided to trade using only money and spot markets. A central banking system is formed as follows.

The central bank issues an amount of money M which it sells for personal IOU notes to be paid back at the end of the economy, that is, at the financial settlement at the start of period $T+1$.

Figure 4



An individual i is permitted to bid any amount of personal credit in order to obtain a share of M. Suppose i bids u^i. Then if $u = \int_{i \in I} u^i$ (where I is the set of all traders), then that individual obtains $s^i = u^i M/u$ (where s^i = the amount of outside money individual i obtains). The equation determining the rate of interest in the economy for loans of T periods is

$$\frac{u}{M} = (1+\rho)^T. \tag{15}$$

Strategically we may view the outside bank as part of the rules of the game. Fixing M is fixing the outside money supply; fixing ρ leaves the outside supply to be determined by the strategic behavior of the traders. In general, if both ρ and M are fixed, credit rationing must be considered.

The initial accounts appear as in Figure 5. The contract u reads, "In return for M obtained at the start of $t = 1$, u will be returned at the start of $t = T+1$. At time t it has a present value of $M(1+\rho)^{t-1}$."

At this point we need to make a decision concerning accounting practices and economic analysis. Should the books be kept on an accrual or cash basis? In an economy without uncertainty and with perfect trust it does not matter. They will both reflect the same economic reality. Otherwise this is not necessarily true.

Figure 5

Outside Bank		Individuals	
u	M	A	u
		M	A

2.3.4. *Failure to Repay and Bankruptcy*

Closely related to accounting practices are practices concerning the rollover and refinancing of loans or other payments due. If it is always possible to refinance, then any failure may be put off until the end of the economy.

In order to make final indebtedness unattractive, there must be penalties for those who fail to settle accounts. The simplest yet quite general rule is to imagine that at the end of the economy positive amounts of money have no utility, but debt has negative utility. Thus we may modify the utility function suggested in (1) to become

$$U^i = \sum_{t=1}^{T} \beta^{t-1} \phi(c_t^i) + \lambda \min[0, D_{T+1}] \tag{16}$$

where

D_{T+1} = final cash balances
λ = a penalty parameter.

We could make the form of the penalty term far more complex, but as we have shown elsewhere qualitatively the same results are obtained (see Shapley and Shubik 1977). Essentially, if credit is granted, then in a game of strategy, rules must be specified to cover the contingencies of failure to repay. Figure 6 shows the approach adopted here.

At the end of the economy the holding of positive amounts of money has no value, as is indicated by the vertical indifference curves. However, ending in debt has negative utility, as is indicated by the curvature of the indifference curves in the negative orthant. The specific form used in (16) is indicated by the straight lines with slope λ in the negative orthant.

Qualitatively, it is simple to state our previous results. If the penalty λ is made high enough, then in the noncooperative game using money no one attempts to take strategic advantage of the possibility of being able to go bankrupt and the resultant noncooperative equilibria include the competitive allocations.

In terms of the special model already described in section 2.1, all we need is that $\lambda > \mu$ and we obtain a noncooperative equilibrium with $u = M$ and $\rho = 0$. This will be considered more elaborately below.

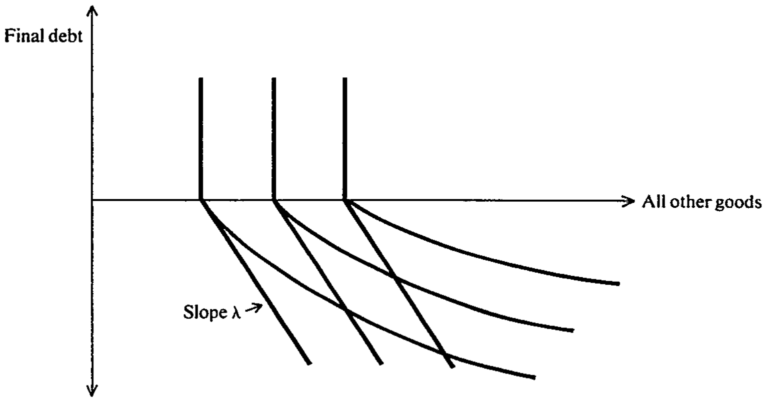
2.3.5. *Inside Banks*

Given that the outside bank has been formed and has supplied M units of outside money to the traders, we now consider the sale of shares in a set B of inside banks. We may save on notation by normalizing the number of shares to one.

Assume that an individual i bids v_j^i for shares of bank j where $\int_{j\in B} v_j^i = v^i \leqslant s^i$ and v_j^i is in outside bank money. Bank j will have a capital of $v_j = \int_{i\in I} v_j^i$ resulting from the sale of its shares.

We assume that the banks stand to lend and accept deposits. We distinguish between lending and depositing. We assume that private individuals with surplus cash do not lend it directly to others. They either hoard it or deposit it with a bank. If no one bids for shares of the inside banks, they do not come into existence.

Figure 6



We defer a discussion of the strategies of the individuals, inside banks, and firms until section 2.4, so the specification of the loan policy is delayed until then.

2.3.6. *The Creation of Firms*

In this formulation, firms are created after the financial system. Even so, there are alternatives in the creation of firms, each of which may have different financial implications in terms of the volume of trade and cash needs of different entities.

Case 1. Firms are created by an exchange of resources for stock.

Case 2. Firms are created by a sale of stock for money, after which money is used to buy resources offered for sale by individuals.

Case 3. Firms are created by a sale of stock for money, after which the money is used to buy resources. However, individuals must offer all resources for sale. They are permitted to buy back resources, but every item must pass through a market at least once prior to its consumption.

Although the last case may appear to be the most unrealistic, it has the advantage that all of consumption is monetized. The only barter possible is that which occurs when a firm sells itself its own assets and avoids incurring any actual transaction while doing so.

Assume that individual i bids z_k^i for the shares of firm k, where z_k^i is in outside or inside bank money or a combination of both. We normalize the number of shares of any firm to be one.

Firm k will have a capital of $z_k = \int_{i \in I} z_k^i$.

2.3.7. *Trade, Production, and Consumption*

Given the existence of all institutions, we consider a standard cycle where individuals and firms refinance, then trade in a market for goods for production and consumption.

We assume that (after the first period) the firms will hold all capital which at time t will be divided into goods for sale c_t and for reinvestment I_t. An individual i will bid b_t^i for the product of the firm offered at time t. Thus price will be given by

$$p_t = b_t / c_t.$$

The firms take their capital $k_t - c_t$ and produce new capital

$$k_{t+1} = g(k_t - c_t) = g(I_t).$$

2.3.8. *The Accrual Accounting, Roll-Over Loan, Assessible Stock Finesse*

For purposes of simplification, in our initial study to investigate the money rate of interest, we make assumptions frequently not borne out in reality.

- Firms and banks pay out their profits only at the end of time.
- Loans can always be rolled over.
- All shares are fully assessible, so given the accrual of profits we do not need to worry about interim corporate or bank failure; losses are eventually flowed through to individuals who will have to face up to the penalty of indebtedness at the end of the economy.

2.4. *Strategies of Firms, Banks, and Individuals*

2.4.1. *Firms*

A strategy by a firm consists of buying resources in the first period;[3] producing, and then dividing further capital into that retained for production and that placed on sale for consumption. As money income is received, the firm will deposit it in the bank and let it accrue until the end. After the last period, the firm liquidates and pays out the total to the stockholders while retiring its stock.

In equilibrium, since no firm is large enough to influence the market, a firm will apparently face fixed prices $p_1, p_2, \ldots, p_{T+1}$ and interest rates $\rho_1, \rho_2, \ldots, \rho_T$. Thus a firm starts by borrowing from (depositing in) the bank an amount d_1 and buying capital $I_1 = p_1(A - c_1)$. Then it produces k_2 and divides it into $I_2 = (k_2 - q_2)$ and q_2 for sale. At the start of the third period, the firm banks $\gamma_2 \ldots$. Thus a strategy is described by

$$(d_1, I_1, k_2 - q_2, q_2;\ \gamma_2, k_3 - q_3, q_3;\ \ldots, \gamma_T, k_{T+1})$$

where

$$\gamma_t \leqslant p_t q_t.$$

2.4.2. *Banks*

We may consider several variations for the strategy of a bank. The first and simplest is the fixed passive 100 percent reserves policy. The bank is permitted to lend up to its total capital. This capital may include accrued or received interest. Deposits are essentially brokered costlessly, that is, they are loaned out at the going rate, but the earnings are paid to the depositors.

A second variation includes the possibility of a variable reserve ratio. Thus at any time t the central bank specifies a number r_t such that an inside bank is permitted to lend up to r_t times its total capital. This leverage could be so large that capital requirements are negligible.

If the banks are so small that they cannot influence the rate of interest, in the first instance the money supply cannot be increased faster than by the previous rate of interest. In the second instance the supply of money is varied exogenously by changing the reserve ratio.

A small competitive bank in an environment with no uncertainty will always operate without excess reserves. It will always be loaned up. Thus in both formulations suggested above the banks act passively. They lend all that they have to lend. If equilibrium rates of interest exist, then the demand for the banks' funds equals their supply. (Banking is discussed further in section 4.)

In reality, banks are not necessarily small with respect to their markets, and costs are not constant.

2.4.3. *Individuals*

A strategy by an individual is first to bid for outside money and bank shares; then to borrow, bid for shares in the firms, buy consumption goods, and refinance; and in all subsequent periods to buy consumption goods and refinance. It may be represented by

[3] If we assume a concave production function such as $k_t = \sqrt{\alpha I_{t-1}}$, this implicitly covers the presence of a limited second resource and we are required to consider the sale of the production function in order to balance the books and account for profits.

$$(u, v, e_1, z, b_1, e_2, b_2, \ldots, e_T, b_T)$$

where

u = bid for outside money
v = bid for inside bank shares (a vector)
e_t = borrow from or deposit in inside banks at t
z = bid for shares in firms (a vector)
b_t = bid for consumer goods (a vector).

Since our purpose is limited to the examination of equilibrium points of the capital stock modified competitive equilibrium and associated noncooperative games, the above description of what constitutes strategies is somewhat informal, leaves out a discussion of information condition, and cuts corners on notation. These points have been discussed in detail in Dubey and Shubik 1977a,b.

3. The Capital Stock Modified Competitive Equilibrium

In this section we take a direct extension of the competitive equilibrium model noted in section 2.1, and without going into the detail called for by the noncooperative game we solve a model closely related to that in 2.1 for different growth rates. (See also Dubey and Shubik 1977a,b.) In particular, we consider four cases: no capital stock left over, initial and final capital stock constant, 10 percent growth, and maximum growth.[4]

Using the model of section 2.1 with the modifications indicated below, we calculate four different levels of growth for the case $A = 100$ and $\beta = 1$ (see Table 2).

Table 2

	c_1	c_2	c_3	p_1	p_2	p_3	μ	k_1	k_2	k_3
No ending capital	46.81104	88.0082	152.1228	1	.729311	.554724	.073080	100	145.8616	152.1228
Stationary capital	41.06995	69.6929	83.1272	1	.767658	.702895	.078020	100	153.5318	183.1272
10 percent growth	40.17960	67.2762	76.9800	1	.772809	.722460	.078880	100	154.6800	186.9800
Maximum growth	0	0	0	no price system				100	200.0000	282.8400

The capital stock modified equilibrium conditions have instead of (4) and (5)

$$\max \sqrt{c_1} + \beta\sqrt{c_2} + \beta^2\sqrt{c_3} + \lambda \min \{0,[\Pi/1+\rho - (c_1 + p_2c_2 + p_3c_3)]\} \tag{4'}$$

$$\max A - I_1 + p_2\left(\sqrt{\alpha I_1} - I_2\right) + p_3\left(\sqrt{\alpha I_2} - I_1\right) + p_4\sqrt{\alpha I_3} \tag{5'}$$

where Π in equation (4′) indicates the profit paid to the consumer which in this highly symmetric example is the amount given by equation (5′). The parameter p_4 is set exogenously and determines the growth of the economy.

[4]We could also consider depletion in general.

It is easiest to carry out all calculations in terms of present values. Hence, although in the noncooperative game we consider only spot markets for goods, here we consider p_1, p_2, p_3, p_4 to be the present values of spot prices in the future.

In (4′) Π is divided by $(1+\rho)$ indicating that there is a time lag in the receipt of income. It is easy to see that if $p_4 = 0$ then $\rho = 0$ and there is 0 ending capital which corresponds to the Arrow-Debreu general equilibrium solution.

In addition to equations (4′) and (5′) and equations (6) to (16) we have

$$(9') \qquad \frac{1}{2\sqrt{c_1}} = \lambda$$

$$(10') \qquad \frac{\beta}{2\sqrt{c_2}} = \lambda p_2$$

$$(11') \qquad \frac{\beta^2}{2\sqrt{c_3}} = \lambda p_3$$

$$(12') \qquad \frac{\Pi}{1+\rho} = c_1 + p_2 c_2 + p_3 c_3$$

$$(13') \qquad \frac{p_2\sqrt{\alpha}}{2\sqrt{I_1}} = 1$$

$$(14') \qquad \frac{p_3\sqrt{\alpha}}{2\sqrt{I_2}} = p_2$$

$$(17) \qquad p_3 = \frac{p_4\sqrt{\alpha}}{2\sqrt{I_3}}.$$

For the stationary economy[5] we require $p_4 = p_3 = .702895$ and $\lambda \geqslant .078020$.

3.1. *Monetization of the Economy and Creation of Firms*

In section 2.3 we noted several variations in the creation of firms. Each variant requires a different amount of money (normalizing $p_1 = 1$ in all cases) for the economy. In particular, if individuals are the original owners of all resources, including production technologies (or institutions), then if we form firms by exchanging stock for resources we will use far less money than if stock is sold for money and then money is used to purchase resources.

Although three cases were noted previously, we note a two-by-two breakdown which leads to four cases and includes the situations requiring the least and most money. Table 3 shows the total expenditures of an individual in a stationary economy lasting T time periods under the different arrangements. In each instance the real goods aspects of the economies are the same, but the monetization is different. *Exchange, hold back* uses the least money, and

[5]This formulation is not fully correct. We have not included the income obtained from the profits of the banking sector. We defer this until section 6.

buy stock, sell all uses the most money. Given that some money is used for transactions and a pool or float of noninterest-earning money is formed, for any finite-horizon model (as can be seen from Table 3) there will be a difference in the money interest rate when compared with a different finite-horizon model when the trading technology calls for different amounts of working capital or transaction reserves or bank float in the circuit not earning interest.

We would suspect that over an infinite horizon the differences caused by initial conditions should ameliorate.

The important feature to appreciate, however, is not the details of initial conditions or the trading technology, but the fact that if outside money exists, that is, if money is supplied by the government or any agency beyond private members of the economy, then for a positive rate of interest there must be a drain of money from the economy.

Table 3 is calculated as follows. Given a natural discount rate of β, the optimal stationary state consumption of an individual is $100\beta(2-\beta)$ given initial assets of 200β. The individual must supply the corporation with its raw materials in the amount of $100\beta^2$. The firm must also purchase its production function at price V. This cost will reflect the allocation of profits or quasi-rents earned from a firm with decreasing returns to scale.

Table 3

	Individuals hold back	Individuals sell or exchange all
Exchange	$100\beta^2(2-\beta)\left(\frac{1-\beta^{T-1}}{1-\beta}\right)$	$100(2-\beta)\left(\frac{1-\beta^{T}}{1-\beta}\right)$
Buy stock	$V + 100\beta^2 + 100\beta^2(2-\beta)\left(\frac{1-\beta^{T-1}}{1-\beta}\right)$	$V + 100\beta^2 + 100\beta^2(2-\beta)\left(\frac{1-\beta^{T}}{1-\beta}\right)$

In an exchange of resources for stock, individuals take ownership paper directly for $100\beta^2 + V$. Starting with period 2, the individuals then buy $100\beta(2-\beta)$ from the firm which produces 200β per period. Thus individuals spend

$$(18) \qquad 100\beta(2-\beta)\sum_{t=2}^{T}\beta = 100\beta^2(2-\beta)\left(\frac{1-\beta^{T-1}}{1-\beta}\right).$$

The other three cases are calculated in a similar manner.

3.2. *Policy and Profits of Firms*

The firms are assumed to be run by profit-maximizing automata who consider the maximization of the present value of the expected total income stream including final liquidation value of the firm.

We consider a T time period world with a final settlement and liquidation period at $T+1$. In period $T+1$ an outside agency (the government, god, or referee) announces prices for all leftover resources. In particular, here we assume a price p_{T+1} is announced that motivates the firms to leave 200β at $T+1$. Since production functions are neither destroyed nor augmented in this economy, a price of $V/(1+\rho)^T$ is offered for redemption where ρ is the money rate of interest.

If the prices $p_1, p_2, \ldots, p_{T+1}$ and $V/(1+\rho)^T$ are all regarded as futures prices, then we have two cases representing the present worth of the firm. They correspond to the *hold back* and *sell all* cases. In the original period of formation the owners hold back their first period of consumption rather than place the resources in the firm and buy back consumption. In the second instance all resources go to the firm. Thus in Case 1 (hold back), less is invested and the firm is worth less. In Case 2 (sell all), the firm is worth more.

Case 1. (Hold back)

(19) $$V + 100\beta^2 = 100\beta^2(2-\beta)\left(\frac{1-\beta^{T-1}}{1-\beta}\right) + 200\beta^{T+1} + V/(1+\rho)^T$$

Case 2. (Sell all)

(20) $$V + 200\beta = 100\beta(2-\beta)\left(\frac{1-\beta^{T}}{1-\beta}\right) + 200\beta^{T+1} + V/(1+\rho)^T$$

paid in capital = income + liquidation value

3.3. *Income and Expenditure of Consumer Owners*

Let I and E stand for income and expenditures of a consumer. In a general equilibrium analysis, $I-E = 0$ and there is no worth attached to leftover capital stock. Denoting that by K, we have $K = 0$.

If we have a lag between income and expenditure such that income is obtained after expenditure, we have

(21) $$E - I/(1+\rho) \leqslant E - I = 0 \quad \text{for } \rho \geqslant 0.$$

When $\rho = 0$, the use of money is costless and the equality is satisfied. When $\rho > 0$, instead of (21) we may consider

(22) $$E - I/(1+\rho) = K$$

which links the money rate of interest with the remaining capital stock.

3.4. *The Rate of Interest and the Value of Capital*

Setting aside until section 4 the problem of financing the inside banks and controlling their behavior, we may calculate the money rate of interest and the value of capital as a function of growth in the economy.

We will have four cases of which the least and most money utilization are shown.

(23) Consumer Balance[6] (Exchange/Hold back)

$$100\beta^2(2-\beta)\left(\frac{1-\beta^{T-1}}{1-\beta}\right)\left(\frac{\rho}{1+\rho}\right) = 200\beta^{T+1} + V/(1+\rho)^T$$

[6]An adjustment to account for bank profits must be made for this equation. It is done in section 6.

This is combined with (19) to solve for V and ρ.

(24) Consumer Balance (Buy stock/Sell all)

$$\left[V + 100\beta(2-\beta)\left(\frac{1-\beta^T}{1-\beta}\right) + 100\beta^2\right]\left(\frac{\rho}{1+\rho}\right) = 200\beta^{T+1} + V/(1+\rho)^T$$

This is combined with (20) to solve for V and ρ.

3.4.1. *The Infinite Horizon*

From (23) with (19) and from (24) with (20) we may derive

$$\lim_{T\to\infty} \frac{\rho}{1+\rho}\{(1+\rho)^T - 1\} = \frac{1-\beta}{2-\beta} \tag{25}$$

which for $\beta = 1$ gives $\rho \to 0$ and

$$\lim_{T\to\infty} \frac{1+\rho}{\rho}\left[\frac{1-\rho(1+\rho)^{T-1}}{(1+\rho)^T - 1}\right] = \frac{(2-\beta^2)}{\beta(1-\beta)} \tag{26}$$

which for $\beta = 1$ gives $\rho \to 0$. Thus for a stationary state with no natural discount rate we have for the infinite horizon a money rate of interest of zero and constant money prices.

For $\beta = 0$, only $T = 1$ is relevant. If, however, $0 < \beta < 1$, then ρ approaches zero as the horizon lengthens. For example, for $T = 19$, $\rho = 1$ percent; for $T = 397$, $\rho = .1$ percent; for $T = 6190$, $\rho = .01$ percent when $\beta > 0$, given the exchange/hold back case.

3.4.2. *The Finite Horizon*

Using the data displayed in Table 2 for four levels of growth over three periods, we calculate the values for ρ and V given the most monetized economy, that is, one in which shares are bought and all trade is monetized.

As we see from Table 4, the interest rate for zero ending capital is zero, for stationary growth is high, and for 10 percent growth is higher. Maximum growth calls for no consumption and an essentially unboundedly high interest rate.

In the case of 10 percent growth, the redemption price of the firm drops from V to $.95346V / (1+\rho)^3$ where the first term reflects the decline in mar-

Table 4

	ρ	$\hat{p}_1c_1$	$\hat{p}_2c_2$	$\hat{p}_3c_3$	I_1	I_2	I_3	λ	p_1	V
No ending capital	0	46.81104	64.18535	84.38571	53.18896	57.85332	0	.073080	0	142.194
Stationary capital	.4755	41.06995	78.93971	124.48809	58.93005	83.83891	100.0	.078020	.702895	178.931
10 percent growth	.5180	40.17960	78.92330	128.15490	59.82040	87.40380	110.0	.078880	.757722	183.632
Maximum growth	—	—	—	—	100	200	282.8	—	—	—

ginal productivity at level 110 compared with 100.

The stationary and 10 percent growth calculations can be made from the data of Table 2, giving

$$V + 100 = 223.2875 + V/(1+\rho)^3 \tag{27}$$

and

$$\frac{\rho}{1+\rho}[V + 211.87] = 70.2895 + V/(1+\rho)^3 \tag{28}$$

for the stationary economy and

$$V + 100 = 231.1352 + .95346V/(1+\rho)^3 \tag{29}$$

and

$$\frac{\rho}{1+\rho}[V + 207.6064] = 83.3492 + .95346V/(1+\rho)^3 \tag{30}$$

for 10 percent growth.

In this model unbounded growth is not possible. Even without consumption, investment cannot exceed 400, or where $z^2 = 20\sqrt{z}$.

4. Outside and Inside Banks

4.1. *Creation of Inside Banks*

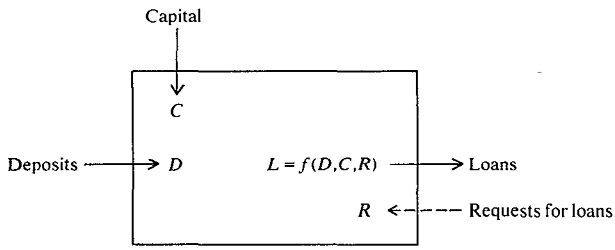
In a one-period economy which uses fiat money, an outside bank or agency can fix an arbitrary amount which will suffice to finance all transactions. If, however, we wish to consider a multiperiod economy using fiat money, then in order for the appropriate price ratios to exist between periods the supply of money used in transactions may vary from period to period.

If the money rate of interest $\rho = 0$, then there is no problem in having an outside bank issue enough money for any finite number of periods. At zero interest rate there is no distinction between hoarding and saving, so if M units of fiat money are issued they will only all be actively used at the period of maximum transactions. At other times, part of the money supply will be in hoard or inactive balances.

If the economy has a money interest rate greater than zero, then the amount of money is not automatically conserved within the system. A mechanism for the variation of the money supply is called for. The creation of an inside banking system (that is, a privately owned system) serves to enable intertemporal exchanges to take place by borrowing, saving, and lending.

It is important to note the institutional aspect of intermediation which differentiates saving from lending. In actuality firms extend credit to each other, friends lend each other money directly, credit is manufactured informally, and much trade involves barter or informal nonmonetary quid pro quo arrangements. The simplest model to stress the properties of monetary trade rules out the granting of credit between individuals and considers no futures markets except for money. Thus banks are given a special role, as indicated in Figure 7. In particular, individual deposits are aggregated. Loans and deposits are between individuals and the bank, thus aggregation of saving and disaggregation into lending takes place. A loan will depend on the aggregate of deposits of the bank, its capital, and the specifics of the demands by those requesting the loans.

Figure 7



Before we discuss how the money supply is varied (see section 4.2 below), we consider the earnings of the inside banks and the price paid for their shares. As has already been indicated in Tables 3 and 4, we have been able to determine a money interest rate and value of capital consistent with the individual consumers maximizing subject to a budget involving transaction costs and the firms maximizing in such a way that the present value of the firm precisely equals what is paid for its stock. We may write an extra equation for the inside bank on the basis that its shares must equal the present value of all earnings plus liquidation. Denote the amount paid for inside bank shares by B. Then for the two extreme cases from Table 3 involving the least and the largest money requirements we have for the stationary state

$$(31)\qquad B = \frac{\rho}{(1+\rho)^2} \sum_{t=2}^{T} p_t c_t + \frac{B}{(1+\rho)^T}$$

and

$$(32)\qquad B = \frac{\rho}{(1+\rho)} \left\{ \sum_{t=1}^{T} p_t c_t + V + I_1 \right\} + \frac{B}{(1+\rho)^T}\,.$$

In the economy which uses the least money, the first borrowing takes place during the second period. Thus the first income earned is discounted by $(1+\rho)^2$. In section 6 we utilize these extra conditions to evaluate the worth of bank shares.

It must be noted that all of the outside money can be regarded as being used immediately to buy the shares of the inside banks without any loss of generality. This is because the earnings from bank shares and the outside interest rate must be the same in an economy with no uncertainty and no frictions or barriers to borrowing.

4.2. *Problems With Competitive Banking and the Money Supply*

In the model described above, the firms use the money capital raised to buy resources. The banks, however, make loans which do not necessarily equal the capital of the bank. Furthermore, the banks do not invest their idle funds. A natural question to ask is, What controls or information must be supplied for the functioning of a competitive banking system? We may even question whether it is possible to have the inside banks competitively determine the money supply and/or the rate of interest efficiently.[7]

The system we have solved in the examples has the government specify the outside money supply, the salvage value for goods, a redemption rate for firms,[8] and the bankruptcy penalty.[9] The money rate of interest is determined by the bidding for outside money [see equation (15)].

Given the outside rate of interest and liquidation values, competition by individuals and/or firms will determine the quantity of inside money issued by a passive banking system with unlimited rights to create money, but only on request.

At time t given ρ_t, individuals can determine the amount D_t of loans they would like, where D_t is a function of ρ_t and the liquidation conditions. If banks can issue on demand and are required to both accept deposits and make loans at the specified rate ρ_t, then supply always equals demand. Furthermore, the payout on deposits cannot exceed income from loans since the bank shares would not be sold otherwise.

The system described has passive nonstrategic banks. Could we have an efficient banking system with price (that is, interest rate) or quantity competition? If as a first approximation we assume that banking is a constant returns to scale industry[10] (say, costs are zero), then price competition in general cannot determine an efficient interest rate (see section 4.3) unless the quantity of money is exogenously controlled each period. This follows essentially from the Bertrand-Edgeworth duopoly (or oligopoly) model. (See Shubik 1959.) If the central bank or referee sets the rules on what a private bank can loan according to some reserve ratio formula such as a function of capital and deposits, then all banks will always be loaned up if the interest rate is greater than zero. But the example in section 3.4 provides an instance where the reserve ratio formula needed to avoid excess reserves at the appropriate interest rate would be essentially as complex as fitting a function to a series of T points; that is, it would be equivalent to announcing a money supply constraint each period.

If a strategy by a bank is to announce an amount of loanable funds, L less than or equal to some bound, then we would have interest rates determined by

$$1+\rho = D/L$$

where D is the amount of money due one period hence offered for L. If there are many banks, then limiting Cournot oligopoly behavior gives rise to the same problem as with the price model; that is, banks will not have excess reserves at positive rates of interest.

In summary, it is suggested here that we cannot have a competitive[11]

[7]For a discussion of what is meant by efficiency, see section 4.3.

[8]For production functions homogeneous of order 1, $V = 0$.

[9]The full significance of the bankruptcy penalty has been discussed in Dubey and Shubik 1977a and is not central to the discussion here.

[10]With increasing returns to scale, competition would be eliminated anyhow.

[11]In actuality there may be room for competition in terms of services and cost efficiency.

inside banking system which determines both the quantity of money and the interest rate.

In our model competitive bidding for outside money determines the (long-run) interest rate for outside money. Competitive bidding for shares in inside banks, together with the rule that an inside bank must accept deposits and pay the going rate on them, guarantees that the outside money rate of interest is greater than or equal to the inside money rate of interest. Hence we have a system in which the outside rate of interest is determined competitively and is linked to inside bank earnings and the inside banks are a passive mechanism constrained to fill loans and accept deposits. Thus the money supply is varied.

4.3. *On Modified Pareto Optimality*

In section 4.2 and elsewhere we have referred to efficient banking and efficient allocations. For the classical competitive equilibrium model of T time periods with no liquidation values, our definition remains as usual. When there is an outside agency which has announced liquidation conditions and either an interest rate or money supply conditions for each period, then by *efficient allocations* we mean the Pareto optimal set of this constrained feasible set of outcomes.

4.4. *Multistage Financing and Working Capital*

In section 3.4 several models of the economy with different transaction needs for money were considered. It was observed that if outside money is issued and a positive interest rate exists then the economy must be cash consuming.

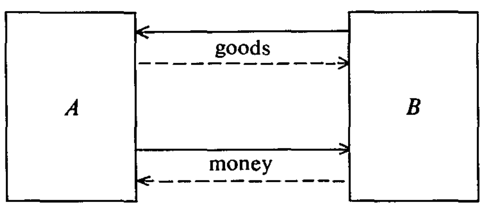
In different economies the technological and strategic features of trade and production determine the needs for working capital, the size of the float and lags and biases in the pattern of payments. Some individuals are paid in advance; others wait for months. The velocity of payments is clearly partly influenced by the rate of interest; it is also determined by technology and custom.

Some individuals may try to live off the float, indulge in check kiting or strategic manipulation of the paying of bills. All of these activities would have quantitative but not qualitative effects on the models in 3.4.

Working capital needs are influenced both by the number of stages in production and by the level of integration. Thus, as is suggested by Figure 8, the consolidation of two firms in a buyer-seller relationship is tantamount to introducing a barter exchange.

Horizontal integration merely poses minor accounting problems (if com-

Figure 8



plex taxation is not present), in that the payments pattern between A and B may be biased in favor of one or the other. But when they are considered as a whole, the possible nonsymmetry is no longer visible.

The more important feature is reflected in the possibility that as the number of stages of production for a given output are increased—if we fix the price of the final output (which for a single product or aggregate we can do without loss of generality) for purposes of comparison—the economy with many stages of production will require more capital and hence more money to finance it than the one with fewer stages.

We illustrate the multistage financing by a simple extension of the example given in section 3. In particular, instead of having only a single production function of the form $x_{t+1} = 20\sqrt{x_t}$ we introduce an intermediate good such that before the product of the first stage of production can be used for consumption or production it must be stored (or otherwise processed) for a period. We have

$$y_{t+1} = 20\sqrt{x_t}$$

followed by

$$x_{t+2} = y_{t+1}.$$

It is straightforward to check that all the previous conditions for a stationary state are the same except for the introduction of extra initial conditions and variables for the intermediate good. In particular, the solution is illustrated for $\beta = 1$ and $T = 3$. The new initial conditions require $A_1 = 200$ and $A_2 = 200$, where A_1 is the initial supply of output from the first stage and A_2 the initial supply of output from the second stage. Thus, although final consumption is the same in this economy as in the previous one, more capital is required. (See Böhm-Bawerk 1959 or more easily Lutz 1968 for a simple exposition of Böhm-Bawerk.)

In the start of the economy we can consider the setting up of vertically integrated firms or two sets of firms, one of which sells to the other. If we adopt the former convention, then trade between the two firms is not monetized because it is taken care of by internal accounting. Either convention of forming a set of vertically integrated firms or of forming two sets of firms will yield the same new equations which are modifications of equations (20) and (24).[12]

Given $\beta = 1$, the salvage prices are β^T, β^T, and $V/(1+\rho)^T$. Then we have

$$V + 200\beta + 200\beta = 100\beta\,(2-\beta)\left(\frac{1-\beta^T}{1-\beta}\right) + 200\beta^{T+1} + 200\beta^{T+1} + V/(1+\rho)^T \tag{20'}$$

$$\left[V + 100\beta\,(2-\beta)\left(\frac{1-\beta^T}{1-\beta}\right) + 100\beta^2 + 100\beta^2\right]\left(\frac{\rho}{1+\rho}\right) = 200\beta^{T+1} + 200\beta^{T+1} + V/(1+\rho)^T. \tag{24'}$$

[12]The residual values of the firms $V_1 + V_2 = V$ the value of the integrated firm. In this example $V_2 = 0$.

Setting $T = 3$ we reduce (20′) and (24′) to

$$4(1+\rho)^3(\rho-1) = 4\rho - 1.$$

Hence, $\rho = 141.3$ percent and $V = 322.989$.
For $T = 70$ we obtain $\rho = 4.83$ percent and $V = 7267.545$.

5. Multigenerational Economies

5.1. *Finite and Infinite Stationary Economies*

In the previous sections we have examined finite economies of arbitrary length with one generation. We may extend our analysis to consider overlapping generations. The simplest model is as shown in Figure 1b. Suppose that each generation is active for T periods after which it dies and settles accounts with society at the start of period $T+1$. Society passes on the capital to the new individuals at the start of that period, and this process is continued indefinitely.

Although the process may continue indefinitely, each individual has a finite life and hence faces a final settlement date.

The example from the previous sections is continued. Suppose

$$A = 200 \qquad \text{with } \beta = 1$$

$$T = 70$$

$$\lambda = \mu = 1$$

$$p_{70g+1} = 1$$

V_g valued at $V_g / (1+\rho)^{70}$.

From equations (20) and (24) we obtain

$$V + 200 = 7000 + 200 + V/(1+\rho)^{70}$$

or

$$V = 7000\left[\frac{(1+\rho)^{70}}{(1+\rho)^{70} - 1}\right] \tag{33}$$

and

$$\frac{\rho}{1+\rho}\{V + 7100\} = 200 + \frac{V}{(1+\rho)^{70}} \tag{34}$$

giving

$$(1+\rho)^{70}\{139\rho - 2\} = 139\rho + 68.$$

Hence

$$\rho = 4.276 \text{ percent and } V = 7394.46.$$

Since the corporations may have indefinite life, we may interpret the first generation as one in which the corporations are set up, but at $t = 71$, instead of considering the liquidation of the inside banks and the corporations, we may consider that inside bank shares and corporate shares are redeemed and immediately redistributed to the new generation.

From (32) we obtain $B = 627.85$.

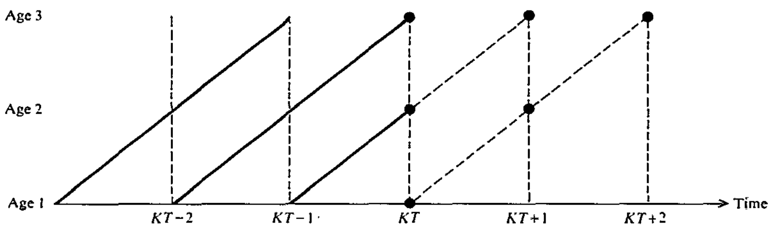
Unlike our previous observation in section 3.4.2, that as the horizon of the economy lengthens the rate of interest approaches zero, here we have a stationary state with a positive rate determined by the life of a single generation. Although all physical quantities are stationary, the prices inflate constantly given $\beta = 1$; as β decreases, we may verify that we switch from inflation to deflation.

We may observe two somewhat different types of stationarity. One is *full* stationarity, as shown above, where each period is a replica of the last. The other is *periodic* stationarity, as in Tables 2 and 4, where initial conditions are reproduced after T periods.

If we consider K generations, we do not need to supply liquidation conditions for each generation. If we specify conditions at $KT+1$, the other values are determined by the requirement that the books balance every T periods.

A somewhat more satisfactory model of many generations has them overlapping as is shown in Figure 1a or in the Lexis diagram shown in Figure 9. We can see a stationary population in which each individual lives three periods. At the end of time at KT, one-third of the population has just been born, one-third is middle-aged, and one-third is about to die. Thus, in order to fully specify terminal conditions, we must take a generation for full liquidation. What happens at the six points denoted by dots in Figure 9 must be specified. (Similarly, for full specification of the model, initial conditions for generations 1, -1, and -2 must be given, that is, conditions for those alive before the start of time and for those just born.)

Figure 9



5.2. *Pareto Optimality*

As long as births are exogenous to the model and the fixing of the terminal conditions are regarded as exogenous, then the comments made in section 4.3 apply to this multigenerational model. Except for the requirements of the ending conditions, there is no essential difference between the multigenerational economy and the standard closed system with production and trade. An economy in which individuals live for T periods, which is active for

K full generations, will have $T(K+2) - 2$ cohorts [that is, individuals born at $-(K-1)$, $-(K-2)$,... up until $TK + (K-1)$]. Thus an imputation is a vector of $T(K+2) - 2$ components, and the usual definition of Pareto optimality can be applied.

If births are considered to be in any way endogenous to the model, then the definition of optimality depends delicately upon the assumptions made concerning the strategic control one generation has upon others.

5.3. *The Interlinking Mechanisms*

In the models of section 5.1, the outside bank or state provides the linkage between otherwise completely selfish generations. The money loss experienced by each generation is enough to pay for the capital stock that the next generation starts with.

When we view this process strategically, we may imagine that the population alive at $t = 1$ owns the original resources, bids for outside money, and sets up the inside banks and firms. However, from then on the state transfers the shares redeemed from the dead to the newly born.

We could modify this model to take into account the possibility of intergenerational concern. Thus we could attach a utility to the leaving of bequests. The form of the desired bequest and the way it is transferred may both influence the economic outcome. In particular, we need to modify the utility function to show intergenerational concern. There are several different relatively natural assumptions which can be made.

- Generation A is concerned with generation B's enjoyment of what it gets.
- A wishes to leave B a specific array of goods.
- A's utility depends on leaving B a sum of money.
- A's utility depends on leaving B a sum of money corrected by an inflation/deflation index.

The first convention leads to an unnecessary and not overly economically relevant complication in interlinked utilities. The second assumption, that individuals desire to leave bequests manifested in arrays of goods, may have some limited merits in the leaving of family heirlooms. But the last two assumptions appear to be far more reasonable in terms of the coding of information and simplicity. Furthermore, much of the intergenerational transfer is not primarily economic; value systems, home education, tastes, political views, and the like remind us how limited pure economic views of life are.

Let us consider as our simplest model one in which the first generation attributes θ percent of the money value of its initial assets to inheritance and $(1-\theta)$ percent to the state. Suppose some individuals wish to leave the next generation enough to be in real terms as well off or k percent better off than themselves, taking into account what will be received from the state.

For purposes of illustration, we select essentially the simplest functional forms we can without trivializing the example. We modify (4) to become

$$\sum_{t=1}^{T} \beta^{t-1} \sqrt{x_t} + \lambda \min [0, D_{T+1}] + \omega \max [0, h(D_{T+1})] \tag{4''}$$

where λ is a penalty against debt and ω is a parameter reflecting the desirability of leaving an inheritance.

Two problems must be faced. The first concerns sequencing. Can an individual both end in debt and leave an inheritance? Or does the payment of debt take precedence over the inheritance? Although on occasion assets are siphoned off, leaving creditors unpaid, here we assume creditors are paid first. The second problem concerns the precise form of $h(D_{T+1})$. In fact, monetary intergenerational transfers (except possibly for the very rich) do not appear to be as important as the transfers which take place during life (Guthrie 1963).

Consider the endowment of the first generation. In the example of section 3 this amounts to $A+V$ at the market price at the start, of which from the initial conditions $\theta(A+V)$ is from inheritance and the remainder from the state. Suppose that the individuals wish to leave the next generation as well off as their generation. The specific strength and nature of this desire is reflected in $h(D_{T+1})$. For example

$$h(D_{T+1}) = \theta(A+V)\left[1 - \left(\frac{D_{T+1} - (A+V)}{(A+V)}\right)^2\right]$$

indicates that leaving too large an inheritance is considered as unfavorable as leaving too small an inheritance. For purposes of calculating an example, we use

$$h(D_{T+1}) = \min\,[D_{T+1}, \theta(A+V)].$$

Thus (4″) becomes

$$(4''') \qquad \sum_{t=1}^{T} \beta^{t-1}\sqrt{x_t} + \lambda \min\,[0, D_{T+1}] + \omega \max\left\{0, \min\,[D_{T+1}, \theta(A+V)]\right\}$$

which indicates that utility increases linearly up to the point of funding the next generation for the private contribution to the same standard of living. Beyond that level no extra benefits accrue to the donor.

We now return to the original example using the stationary state with $T = 70$ as in section 5.1. Equation (35) is now modified, giving[13]

$$(36) \qquad (1+\rho)^{70}\{[141 - 2(1-\theta)\rho - 2(1-\theta)]\} = 68(1-\theta) + [141 - 2(1-\theta)]\rho.$$

For $\theta = 0$ we obtain (35). For $\theta = 1$ this becomes

$$(1+\rho)^{70}\{141\rho\} = 141\rho.$$

Hence $\rho = 0$. For $\beta = 1$, V is not well defined. For $\theta = 1/2$ we obtain

$$(37) \qquad (1+\rho)^{70}\{140\rho - 1\} = 34 + 140\rho$$

$\rho = 3.39$ percent, and $V = 7751.422$.

Comparing this with the result from (35), we see that the more individuals are innately willing to leave to the next generation, the lower will be the rate of interest.

[13]Given $\lambda = \omega = 1$ and $p_T = 1$, it can be checked that equilibrium is as before, with the difference being a smaller component by the government.

6. Taxation and Corporate Reality

6.1. *Central Bank Profits, Taxes, and Interest*

In the models presented here, there is a money drain from the system as a whole which is measured by the profits of the outside or central bank. The system loses $M(1+\rho)^T - M$ to the central bank, where M is the outside money issue and ρ the money rate of interest. This loss is made up or balanced at period $T+1$ by the purchase of residual assets at the final settlement date. Thus the rate of interest is essentially a money transactions tax or, over the full life of an individual, an income tax.

Money flows out of the system, but in order to obey conservation it must be balanced. This is done with the balancing of the worth of final assets. In the Arrow-Debreu general equilibrium, the value of assets is zero and there is either no money or, if the model is interpreted as having money, the rate of interest is zero or is not determined.

All other things being equal, taxation could be used directly as an alternative to having all capital financed through central bank profits or through altruism. The effect of an income tax will be to introduce a different leak in the economy and hence lower the rate of interest needed to produce the financing of the capital goods.

We reconsider the example of section 3, taking into full account inside and outside banking profits and income and capital gains taxes. Before the original equations are modified, we must digress into a consideration of accounting conventions and profits.

6.2. *Accounting, Profits, and Taxes*

In many actual modern economies the individuals and enterprises are confronted with an array of different taxes, and the tax bills are highly dependent on accounting conventions and law. (See, for example, U.S. Internal Revenue Code of 1954, June 1, 1976, edition.) In particular, key considerations are the definition of a taxable event, profits, and income.

In particular, in the United States there are at least three critical tax distinctions: individual, corporate, and capital gains.

According to economic theory, the income of an individual or corporation in an economy which lasts for T periods is given by the total discounted income stream and liquidation value of the investment. To the tax collector, income is usually defined over a period of a year, and in general for individuals, income is an actual cash flow to the individual caused by transactions. For the firm, income will frequently be on an accrual basis, thus not matching cash. Furthermore, for both firms and individuals, unpaid profits, unrealized capital gains, and other forms of increase in wealth not matched by actual transactions are not recorded as income.

The microeconomic theorist wishing to reconcile microeconomic theorizing with macroeconomic approaches and with corporate reality must appreciate the full implications of the relationship among economic, accounting, and legal definitions of items such as income. With the three different types of taxes noted above and with interest payments being deductible from gross income as an expense but dividends not being deductible, the conditions for Modigliani-Miller equivalence (Modigliani-Miller 1958) between debt and equity finance do not hold even without uncertainty.

In particular, it is of interest to note that the concept that the firm should maximize the present value of profits for the good of its stockholders is not necessarily reasonable, given different tax structures. The timing of the booking of profits and the paying of dividends to individuals may be critical to the welfare of the stockholders.

We must return to the basic optimization equations to appreciate this. We study the stationary state with a government which intends to finance the stationary state primarily through taxation and slightly through central bank earnings. We consider three taxes, all proportional to the sum taxed:

η_1 = individual income tax
η_2 = capital gains tax
η_3 = corporate income tax.

Because all taxes are proportional, in spite of the usual convention that they are not indexed (that is, they are paid on spot not inflation corrected incomes or capital gains), we can work with present values (futures prices) without distortion.

We now set up the new full (nonatomic) noncooperative game. We need to be explicit about extra rules concerning taxation planning.

Individuals bid u for the supply of outside money M. The u is to be paid back at time $T+1$. Thus

$$\frac{u}{M} = (1+\rho)^T.$$

Inside banks are formed. Since they are essentially strategic dummies, numbers do not matter. The individuals using inside money $v \leqslant M$ buy shares of the inside banks. (We can assume there is only one without loss of generality.)

If the government is controlling the economy for a stationary state, then if B is the capital of the bank, $B/(1+\rho)^T$ will be offered as the salvage value of the bank's paid in capital (in present value terms).

In competition, if E_t is the present value of earnings during t, then the value of bank shares will be

$$B = \sum_{t=1}^{T} E_t(1-\eta_3) + B/(1+\rho)^T \tag{38}$$

$$\text{after-tax income} \quad + \quad \text{return of capital.}$$

The bank has no capital gains tax to pay when it liquidates.[14] Furthermore, if it has to book the payment of interest on an accrual basis, it has no strategic freedom in reporting income. Hence we may write (38) to evaluate the bank shares. We note that the value is directly influenced by the level of corporate taxes.

After the inside banking system exists, individuals borrow and buy shares in corporations which are being formed. The different possibilities in equity financing have been already noted in section 3.1 (debt finance was not discussed). However, the number of firms to be formed was not noted. Given that the assumption was made that production functions are individually

[14]This is true if inflation is less than the money rate of interest.

owned, an upper bound for the number of firms is one per consumer (leaving out layers of intermediate good firms as in 4.4). For purposes of the argument here, as long as there are many firms, the relative densities of consumers and firms do not matter. The simplest is to consider the same densities.

In our analysis so far we have not discussed debt finance. It has two important features: one associated with taxation, the other with uncertainty. We first consider equity finance, then comment on the difference made by considering debt.

As can be seen from equation (39), a firm needs to raise money to buy its initial assets plus its production technology. However, its income will be taxed, so its present value must reflect this. Hence

$$V + 200\beta = 100\beta(2-\beta)\left(\frac{1-\beta^T}{1-\beta}\right)(1-\eta_3) + 200\beta^{T+1} + V/(1+\rho)^T. \tag{39}$$

If the stationary state exists with taxation, then consumer spending will be

$$V + 200\beta + 100(2-\beta)\beta\left(\frac{1-\beta^T}{1-\beta}\right). \tag{40}$$

Offsetting the consumer's spending is income after taxes

$$\frac{1}{1+\rho}\left[V + 200\beta + 100\beta(2-\beta)\left(\frac{1-\beta^T}{1-\beta}\right)(1-\eta_2)(1-\eta_3)\right] \tag{41}$$

obtained from the firm. The factor $1/(1+\rho)$ indicates that we have attributed the transaction money holding and float loss to the consumer. This has to net out to the consumer because any payments advantage among firms is reflected to the consumer in income.

We must account for consumer income derived from inside banking profits. We may consider the presence of a float between the consumer and the bank, that is, a lag in payments in which the consumer must pay debts promptly but has a lag in payments of income. Alternatively, we may assume that profits are credited instantly to the consumer. Adopting the last convention, if W is borrowed from the bank, then

$$\rho W(1-\eta_3) \tag{42}$$

is paid back as after-tax profits. Hence total consumer income after taxes is

$$\frac{1}{1+\rho}\left[V + 200\beta + 100\beta(2-\beta)\left(\frac{1-\beta^T}{1-\beta}\right)(1-\eta_2)(1-\eta_3)\right] + \rho W(1-\eta_2)(1-\eta_3) \tag{43}$$

plus at the end a return of capital from liquidation of

$$\frac{200\beta^T}{1-\eta_3} + V/(1+\rho)^T + B/(1+\rho)^T. \tag{44}$$

We note that in expression (43) above the consumer has been taxed at the capital gains rate η_2 rather than the individual income tax rate η_1. This follows immediately from the choices of the firm and weak Pareto optimality. As income flows in, if the firm is actually trying to maximize the present value of dividends plus liquidation of residual assets after taxes, it does not matter whether the firm pays dividends or not. If capital gains taxes are lower than income taxes, the firm here should never pay dividends. Instead, earnings will be returned as part of increased equity at the end and will be taxed less. This will be generally true for firms not facing exogenous or strategic uncertainty.

The total debt that must be financed by a consumer is calculated from (40) and (43). The difference between them equals W or

$$W = \frac{\rho(V + 200\beta) + (\rho + \eta_2 + \eta_3 - \eta_2\eta_3)I}{(1+\rho)[1 + \rho(1 + \eta_2)(1 + \eta_3)]} \tag{45}$$

where

$$I = 100\beta(2-\beta)\left(\frac{1 - \beta^T}{1-\beta}\right).$$

Hence the value of the bank shares is given by

$$B = \rho W(1 - \eta_3) + B/(1+\rho)^T$$

or

$$B = \rho W(1 - \eta_3)\left(\frac{(1+\rho)^{70}}{(1+\rho)^{70} - 1}\right). \tag{46}$$

If we let θ be the percentage of the worth of initial assets (in present value terms) that a consumer wishes to leave in her or his estate, then the balancing of the consumer accounts calls for

$$W = (1-\theta)[200\beta^{T+1} + V/(1+\rho)^T + B/(1+\rho)^T]. \tag{47}$$

For simplicity, set

$$k_1 = \rho + \eta_2 + \eta_3 - \eta_2\eta_3$$
$$k_2 = (1 - \eta_2)(1 - \eta_3)$$
$$\omega = 1 - \theta.$$

We may then write (47) for $\beta = 1$, $T = 70$

$$\left[\rho\left(\frac{7000(1 - \eta_3)(1+\rho)^{70}}{[(1+\rho)^{70} - 1]} + 200\right) + 7000k_1\right]\left(1 - \frac{\omega(1 - \eta_3)}{[(1+\rho)^{70} - 1]}\right) \tag{48}$$

$$= \omega\left(\frac{7000(1 - \eta_3)}{(1+\rho)^{70} - 1} + 200\right)(1+\rho)(1 + k_2\rho).$$

A few specific cases are explored for comparison. In particular we consider

(i) $\eta_1 = \eta_2 = \eta_3 = \theta = 0$: no taxes or bequests.
(ii) $\eta_1 = \eta_2 = \eta_3 = 0$ and $\theta = .5$ and $.9$: no taxes but high bequests.
(iii) $\eta_1 > \eta_2 = .25$, $\eta_3 = .5$, and $\theta = 0$ and $.1$: capital gains of 25 percent, corporate income taxes of 50 percent, and bequests of 0.

For (i) and (ii) we may simplify (48) to

$$\rho[71(1+\rho)^{70} - 36]\left(1 - \frac{\omega\rho}{[(1+\rho)^{70} - 1]}\right) = \omega[(1+\rho)^{70} + 34](1+\rho)^2. \tag{49}$$

For (i), $\rho \simeq 4.33$ percent, $V = 7379.7$, $W = 5989.6$, and $B = 273.42$. For $\theta = .5$, $\rho \simeq 3.4$ percent, $V = 7745.8$; $\theta = .9$, $\rho \simeq 1.816$ percent, $V = 9772.6$. For $\eta_1 > \eta_2 = .25$, $\eta_3 = .5$, $\theta = 0$, $k_1 = \rho + .625$, $k_2 = 1.875$, $\rho \simeq .873$ percent, $V = 15{,}357.4$, $W = 4459.37$, $B = 42.705$.

6.2.1. *On Debt Financing*

The key legal economic distinction caused by debt financing in a world without uncertainty is the possibility that interest payments may be regarded as business expenses whereas dividends are not. Thus, all other things being equal, a firm maximizing its present value will select debt finance and in a world without uncertainty will pay no taxes, having no profits. However, the debt holders will pay income tax. If the firm used equity finance and did not pay dividends, the equity holders would only pay capital gains while the firm might pay income taxes.

6.3. *Private Capital and Social Capital*

It is argued here that the simple individualistic model does not provide us with a description of motivation for the formation of even private capital in an economy. By considering trade in money, a society can use the rate of interest as a means of covering the costs of investment. But there are clearly other considerations and techniques for promoting capital formation. In particular, we have

- The desire to leave bequests—altruism
- Corporate and individual income taxes
- Wealth taxes and capital gains taxes
- Other taxes, such as sales taxes
- Earnings of the central bank via interest
- Government debt.

In actuality, much of the revenues gathered by the government go to pay for the governmental bureaucratic infrastructure which may be regarded as a public good or they go for transfer payments to the old and poor, but not to newborn capitalists.

If economists are willing to limit their scope to economics rather than to the whole of social, political, or other choice, then the problem of efficient economic choice can be well formulated. We take the array of public goods desired by society together with the leftover capital stock at the end of time as a datum. Then efficiency is measured in terms of the feasible set of actions in a finite economy.

The question of how much planning and control is needed by a government to obtain the desired amount of capital goods and public goods must be resolved. We return to this in section 8.

6.4. *Turnover, Velocity of Trade, and Taxes*

As a crude approximation, we might consider the life-span of an individual to be around 70 years. We have used $T = 70$ in several of the examples above. The two basic relations of time span are life cycle to manufacturing cycle and life cycle to tax cycle. Income and capital gains taxes are levied on an annual basis, but much manufacturing has a cycle shorter than a year. Thus if we were to consider an economy with five turnovers per annum or a manufacturing length of ten weeks, the individual would live for 350 manufacturing cycles. This correction to equation (48) would cut the interest rate considerably. As was noted in section 4.4, vertical integration or capital deepening, which is a characteristic of much of modern production, would increase the need for financing and raise the interest rate. A depth of from three to five appears reasonable for an economy with extraction, raw material fabrication, manufacturing, wholesaling, and retailing.

6.5. *Heavy Money and Earnings on Deposits and Bank Reserves*

The rate of return on bank reserves will only coincide with the rate of interest on inside money if there is no variability in money supply. Let the rate of return be δ. Then as we have $M = B$ we may write (in present value terms at the end of the economy)

$$M[(1+\delta)^T - 1] = 200(1+\rho)^T + V + B \tag{50}$$

$$\begin{array}{c} \text{cost of} \\ \text{outside money} \end{array} = \begin{array}{c} \text{end value of} \\ \text{capital stock.} \end{array}$$

For illustration we consider the case with no taxes or bequests, $T = 70$, and $\beta = 1$. This instance was studied in section 6.2, yielding $\rho = 4.33$ percent. Solving (50) for this instance we obtain $\delta = 5.53$ percent. Thus when inside banks can create money, bank reserves yield a higher return than bank money does. Outside money is a *heavy money* compared with inside money.

Institutionally, coin and other currency are issued by the treasury, mint, or central bank, and they circulate without earning interest. In the structure noted here, as long as the name of a private controlled bank were held as prime as the name of the central bank, then there would be no need for any issue of government notes or coin.

The system could function with interest payments made on both bank reserves and demand deposits (that is, money used for transactions). These require a modification of (48) with these two features treated parametrically.

7. On Unbounded Growth

Phelps (1966), Solow (1963), Koopmans (1977), and many others have been concerned with economies with unbounded growth, economies in which population grows along with other resources. Ramsey (1928), Koopmans (1960, 1972), and others have been concerned with optimality and ethical considerations about the treatment of future generations. It is suggested here that if the assumption of exogenous population growth is made and financial rules of the game are provided for the economy, then Pareto optimality for a series of generations can be well defined in the conventional way.

We continue our example, introducing labor and population growth explicitly. As before, we assume $A_1 = 200$; we add $L_1 = 100$ as initial population with an exogenous growth rate of n. Thus

$$L_t = 100(1+n)^{t-1}. \tag{51}$$

The full production function becomes

$$z_{t+1} = 2\sqrt{k_t L_t} \tag{52}$$

or

$$z_{t+1} = 20\sqrt{k_t (1+n)^{t-1}}.$$

We modify the starting stock to be $z_1 = 200(1+n)^{-2}$ in order to obtain the golden rule growth where $k_1 = 100(1+n)^{-2}$ and initial per capita consumption is $(1+n)^{-2}$.

Modifying equation (39) we obtain

$$V + \frac{200\beta}{(1+n)^2} = \frac{100\beta(2-\beta)}{(1+n)^2}\left(\frac{1-[\beta(1+n)]^T}{1-\beta(1+n)}\right) + 200\beta\frac{[(1+n)\beta]^T}{(1+n)^2} + V/(1+\rho)^T \tag{53}$$

and modifying (47) we obtain

$$\frac{\rho}{(1+\rho)^2}\left[V + \frac{200\beta}{(1+n)^2} + \frac{100\beta(2-\beta)}{(1+n)^2}\left(\frac{1-[\beta(1+n)]^T}{1-\beta(1+n)}\right)\right] \tag{54}$$

$$= \frac{200\beta}{(1+n)^2}[(1+n)\beta]^T + V/(1+\rho)^T + B/(1+\rho)^T.$$

For $T \to \infty$ we obtain

$$\frac{2\rho}{(1+\rho)^2}\left[\frac{200\beta}{(1+n)^2} + \frac{100\beta(2-\beta)}{\beta n(1+n)^2}\right] = \frac{200\beta}{(1+n)^2}$$

or

$$\frac{\rho}{(1+\rho)^2}\left[2 + \frac{2-\beta}{\beta n}\right] = 1$$

or

$$\rho^2 - \left[\frac{2-\beta}{\beta n}\right]\rho + 1 = 0. \tag{55}$$

We need $\beta(1+n) \geqslant 1$ to avoid some basic problems in the interpretation of the motivation of the stationary state.

Solving for $\beta = 1$ and $n = .02$ and $.05$ we obtain

$$\rho = .020008 \text{ and } .050126.$$

For $\beta = .98$ and $n = .04$, we have

$$\rho = .038488.$$

We see that the money rate of interest is not quite the same as the growth rate but for $\beta = 1$ is slightly higher.[15]

8. Concluding Remarks

In many ways, many of the basic problems involving money also involve uncertainty. In spite of the attractiveness of immediately examining money in the context of uncertainty, it has been suggested in the analysis above that money and the rate of interest can be integrated into economic analysis without uncertainty or even a natural time discount. In doing so, a finite-horizon model for a multigenerational economy can be specified as a non-cooperative game.

The extension of the competitive market to include capital stock provides a determination of the money rate of interest and a transaction demand for money. Liquidity preference and a structure of rates of interest call for the consideration of uncertainty. This is not dealt with here.

8.1. *Many Capital Goods and Societal Goals*

The examples presented in this paper have one capital good which is also the consumer good. It is conjectured that for any number of goods, given that technology and population size is fixed and that all consumption processes involve the use of the individual's time and no better than constant returns hold for production processes, then for $0 < \beta \leq 1$ an upper bound stationary state exists. This is defined in the sense that the referee selects an initial vector of resources, A, and a vector of prices to be paid for all resources at period $T+1$, such that the system at $T+1$ will produce A.

It is further assumed that the life-span and preferences of each generation are identical.

From a practical point of view the calculation of all ending prices in an ongoing economy is out of the question in a noncentralized economy. The motivation for production provided by a wealth-maximizing enterprise is not well defined in actuality without specifying expectations of future prices, demands, and interest rates. The larger the governmental sector and the more clearly enunciated its policy is, the easier and more accurate will be the estimating of the future.

Leaving aside technological change and shifts in taste, then the operational equivalent of the models in the previous sections is that of a society in which individuals believe that policy is designed for a specific level of growth.

8.2. *The Interest Rate as a Control Variable*

The influence of the central bank on the rate of interest is one of the key

[15]It is also of interest to note that in this model with taxes individuals will pay income taxes on labor sold. They will not be able to shelter all of their income in capital gains.

controls in monetary policy. A lowering of the rate of interest in general will increase the demand for bank money and stimulate investment. Yet in the model suggested here the growth rate and rate of interest go in the same direction.

In the models suggested, once the banking system is erected and prices for ending capital given, the interest rate is endogenously determined. If instead we fix it as a control variable, this does not provide enough guidance to the economy to determine prices for ending capital.

8.3. *Statics, Laissez-Faire, and Institutions*

It is suggested here that an attempt to introduce residual capital stock into the static general equilibrium model of an economy immediately raises many basic problems concerning the design and limits of competitive economic mechanisms for an ongoing society. Specifically, unless considerable individual altruism is postulated, the goals and guidance for overall capital formation appear to lie outside of the competitive economy. Furthermore, in a society which uses money it does not appear to be possible to determine both the rate of interest and the money supply competitively.

A passive banking system can be used to provide a variable money supply.

8.4. *A Note on Methodology*

The analysis here is based on two basic premises concerning methodology.

First, a model purporting to explain the functioning of an economy with competitive elements needs to be explicitly *game theoretic* in structure. The discipline called for in fully formulating a well-defined game of strategy forces attention on rules of the game such as methods of payment and bankruptcy rules which have direct counterparts in economic and financial life. The noncooperative equilibrium solution is a far more general solution concept than is the competitive equilibrium of Walrasian analysis. The former contains the latter as a special case.

The second point concerns the role of *gaming*, the construction of playable games which can be used for experimentation. The economy is part of our society and is run by individuals acting through institutions. These have structure and rules. The design of an economic model as a playable game forces on the designer a degree of care and specificity that offers a virtually automatic test of the completeness, consistency, and complexity of the model proposed.

The models presented here do not fully meet the two stringent conditions noted above. The decision of bankruptcy and positions of disequilibrium is skimpy. Yet the attempt to convert the models into playable games appears to be a fruitful approach to the eventual construction of an economic dynamics.