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Cooperative Theory with Incomplete Information*

Beth Allen

Federal Reserve Bank of Minneapolis
and University of Minnesota

ABSTRACT

This paper surveys cooperative game theory when players have incomplete or asymmetric information, especially when the *TU* and *NTU* games are derived from economic models. First some results relating balanced games and markets are summarized, including theorems guaranteeing that the core is nonempty. Then the basic pure exchange economy is extended to include asymmetric information. The possibilities for such models to generate cooperative games are examined. Here the core is emphasized as a solution, and criteria are given for its nonemptiness. Finally, an alternative approach is explored based on Harsanyi's formulation of games with incomplete information.

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1. Introduction

This paper considers the incorporation of uncertainty and information into cooperative game theory. In particular, we examine the cooperative games that arise from economies with asymmetric information. To simplify, we focus on the case of general equilibrium models of perfectly competitive pure exchange economies.

One frequently encounters the opinion that cooperative game theory cannot easily be adapted to include informational considerations. In fact, economists' interest in asymmetric information is sometimes cited as an important reason for the recent emphasis on noncooperative models of strategic behavior, especially in fields such as industrial organization and corporate finance. I disagree with this viewpoint—one *can* put asymmetric information into cooperative games, albeit at the expense of certain complications which may lead to somewhat surprising results.

Since we stress cooperative games that are derived from economies with asymmetric information, we first digress to present a more general, brief survey of the relationships between cooperative games and perfectly competitive exchange economies. After summarizing these results on market games in the next section, we proceed to introduce information in the following section. Section 4 is devoted to Wilson's article on the core with asymmetric information. The derivation of cooperative games from economies with asymmetric information is examined in Section 5, as preparation for analysis of the core and the value in the following two sections. Section 8 concludes by presenting an alternative approach based on Harsanyi's formulation of noncooperative games with incomplete information.

2. Market Games

To fix notation, let N be the (finite) set of traders in the economy (or players in the game), and denote a typical agent by $i \in N = \{1, \dots, n\}$ (where n is the cardinality of the set N). Suppose that the number of commodities present in the economy is the finite positive integer ℓ , and take \mathbb{R}_+^ℓ to be the consumption set of each trader $i \in N$.

Traders are specified by initial endowment vectors and utility functions where, for each $i \in N$, $e_i \in \mathbb{R}_+^\ell$ and $u_i : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$ is a continuous (or, more generally, upper semicontinuous) and concave function representing the preferences of trader $i \in N$. By definition, a *coalition* is a nonempty subset of agents; the *grand coalition* N is a coalition as is any nontrivial collection of agents. Each coalition induces a smaller economy containing only those traders who belong to the coalition; such subeconomies are called *submarkets*, and they induce *subgames*.

An n -player *cooperative game with transferable utility* (or *TU game*) is a function $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$, where 2^N denotes the set of all subsets of $N = \{1, \dots, n\}$. The *TU cooperative game* induced from the pure exchange economy as above, in which each trader $i \in N$ has consumption set \mathbb{R}_+^ℓ , initial endowment $e_i \in \mathbb{R}_+^\ell$, and utility function $u_i : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$ which is assumed to be upper semicontinuous (so that maxima in the definition below are well defined), is given by $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$ and, for all $S \subseteq N$ with $S \neq \emptyset$, $v(S) = \max \left\{ \sum_{i \in S} u_i(x_i) \mid x_i \in \mathbb{R}_+^\ell \text{ for all } i \in S \text{ and } \sum_{i \in S} x_i \leq \sum_{i \in S} e_i \right\}$. [With sufficient monotonicity, the inequality sign can be replaced by an equality.] In words, $v(S)$ is the maximum total utility that the players in S can achieve by redistributing their own resources; if members of coalition S were to pool all of their initial endowments and redistribute these goods, so as to maximize the total utility of the entire coalition S , the resulting sum would equal $v(S)$.

The *TU core* of the n -person *TU game* v is defined to be the set of all payoff vectors $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ such that (1) (w_1, \dots, w_n) is feasible (for N): $\sum_{i \in N} w_i \leq v(N)$, and (2) (w_1, \dots, w_n) is not *blocked* by any coalition: $\sum_{i \in S} w_i \geq v(S)$ for all $S \subseteq N$. Feasibility of $w \in \mathbb{R}^n$ says that (w_1, \dots, w_n) is an *imputation* for v . The second property is sometimes described by the statement that no coalition can improve upon w .

Transferable utility games with nonempty cores [note that the core always exists] can be characterized by a balancedness condition. If N is any finite set, a balanced family \mathcal{B} of subsets of N is $\mathcal{B} \subseteq 2^N$ for which there exist balancing weights $\{\gamma_S\}_{S \in \mathcal{B}}$ with

$\gamma_S \geq 0$ ($\gamma_S \in \mathbb{R}$) such that for each $i \in N$, $\sum_{S \ni i} \gamma_S = 1$. Obvious examples of balanced families include N itself (with weight $\gamma_N = 1$) and $\mathcal{B} = \{\{1\}, \dots, \{n\}\}$ (again with each balancing weight equal to one). For a nontrivial example, consider the two-player coalitions of $N = \{1, 2, 3\}$ and set $\gamma_S = 1/2$ for $S = \{1, 2\}$, $S = \{1, 3\}$, and $S = \{2, 3\}$. A *TU* game on N is said to be *balanced* if, for all balanced collections \mathcal{B} of subsets of N and all collections of associated balancing weights $\{\gamma_S\}_{S \in \mathcal{B}}$, we have $\sum_{S \in \mathcal{B}} \gamma_S v(S) \leq v(N)$. [Note that the left-hand side of this inequality differs from the operation of taking convex combinations in that we could have $\sum_{S \in \mathcal{B}} \gamma_S > 1$.]

Theorem 1. *A finite TU game has a nonempty core if and only if the game is balanced.*

This result was discovered independently by Bondareva (1962) and Shapley (1967). Its proof involves demonstrating that a certain system of linear inequalities has a solution precisely when the constraints defining balancedness are satisfied.

For an example of a game that fails to be balanced, again let $N = \{1, 2, 3\}$, and define (with an obvious abuse of notation) $v(123) = v(12) = v(13) = v(23) = 1$ and $v(S) = 0$ otherwise. Then v is not balanced, since examination of the balanced family of two-player coalitions would require $v(12)/2 + v(13)/2 + v(23)/2 \leq v(123) = 1$, an obvious contradiction. Intuitively, we know that the *TU* core of this game is empty, because any two-player coalition that does not include the best treated player(s) can block any (feasible) imputation. The game describes a situation (“three men and a trunk”) in which three people discover buried treasure which can be removed from the jungle only if at least two individuals carry it.

The Bondareva-Shapley Theorem is of particular interest because it applies to all market games as described above. Moreover, there is an equivalence between games satisfying a stronger balancedness property and those games that can be derived from pure exchange economies satisfying the conditions stated above. A *totally balanced* game is one for which every subgame is balanced.

Theorem 2. *Every market game derived from a finite pure exchange economy in which each trader i has consumption set \mathbb{R}_+^ℓ , initial endowment $e_i \in \mathbb{R}_+^\ell$, and utility function $u_i : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$, which is upper semicontinuous and concave, is totally balanced. Conversely, every n -player totally balanced TU cooperative game can be generated by a pure exchange economy as above with $\ell = n$.*

This result was discovered by Shapley and Shubik (1969) in their classic study of market games. The proof in one direction uses the balancing weights to define feasible allocations that are convex combinations of allocations available to smaller coalitions. Concavity of utilities then implies, by Jensen's inequality, that total utility cannot be forced to decrease in the larger coalition. For the converse, Shapley and Shubik (1969) construct very special economies in which each player's payoff essentially depends only on the player's allocation of one commodity. [Note that the relations between exchange economies and totally balanced games cannot be described by a one-to-one correspondence because the space of n -player games can be identified with a Euclidean space of dimension $2^n - 1$, whereas the space of n -agent exchange economies parameterized by endowments and utilities must be infinite-dimensional. More specifically, changing a trader's utility function off of the compact set of feasible allocations cannot alter the TU game generated by the economy.]

Cooperative games with nontransferable utility (or NTU games) can similarly be derived from economies. Of course, NTU games are preferable in general for economics, as they do not require one to impose the assumption that each agent's preferences are representable by a quasilinear utility function in order to justify the addition of payoffs of different agents. [A quasilinear utility is a function of the form $u(x) + m$, where x can be a vector of goods and m denotes the quantity of a commodity—such as money—in which side payments are made.]

Recall that an NTU cooperative game with player set $N = \{1, \dots, n\}$ is a correspondence $V : 2^N \rightarrow \mathbb{R}^n$ such that, for all $S \subseteq N$, the sets $V(S)$ are nonempty, closed,

and comprehensive [i.e., $V(S) \supseteq V(S) - \mathbb{R}_+^n$], and, moreover, the $V(S)$ sets are *cylinder sets* in that if $u = (u_1, \dots, u_n) \in V(S)$ and if $u' = (u'_1, \dots, u'_n)$ is such that $u_i = u'_i$ for all $i \in S$, then $u' \in V(S)$. I follow the convention that $V(\emptyset) = \mathbb{R}^n$. Define the *projections* of the $V(S)$ sets into the subspace of payoffs for players in S by $V(S)_S = \{u \in V(S) | u_j = 0 \text{ if } j \notin S\}$. Note that $V(N)_N = V(N)$ and $V(\emptyset)_\emptyset = \{0\}$. In addition, for each $S \subseteq N$, $V(S)_S$ generates the cylinder set $V(S)$.

A cooperative game $V : 2^N \rightarrow \mathbb{R}^n$ with nontransferable utility is *balanced* if, for all balanced collections \mathcal{B} on N with associated weights γ_T for $T \in \mathcal{B}$, $V(N) \supseteq \sum_{T \in \mathcal{B}} \gamma_T V(T)_T$. [Note that since $\mathcal{B} = \{N\}$ with $\gamma_N = 1$ is a balanced collection, taking the union over all balanced collections on the right-hand side gives a subset of \mathbb{R}^n which precisely equals $V(N)$.] This definition of balancedness is well suited for economies with concave utilities. An alternative definition, which Billera (1974) terms “*quasibalancedness*,” is weaker. Say that an *NTU* game $V : 2^N \rightarrow \mathbb{R}^n$ is *quasibalanced* if, for all balanced collections \mathcal{B} on N , $\bigcap_{T \in \mathcal{B}} V(T) \subseteq V(N)$. Every balanced game is quasibalanced. As in the case of transferable utility, games with nontransferable utility are said to be *totally (quasi)balanced* if all of their subgames are (quasi)balanced.

The core of an *NTU* game is defined to be the set of feasible imputations that cannot be blocked—or improved upon—by any coalition. Formally, $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ belongs to the *core* of $V : 2^N \rightarrow \mathbb{R}^n$ if and only if $u \in V(N)$ and there does not exist a coalition $S \subseteq N$ (with $S \neq \emptyset$) and a payoff vector $u' = (u'_1, \dots, u'_n) \in V(S)$ such that $u'_i > u_i$ for all $i \in S$. Note that, by definition, the core always exists—every game has a core, although it may be empty. Of course, we are interested in games with nonempty cores. One rationale for the core as a solution concept is the observation that, although not all points in the core may be attractive solutions, whenever the core is nonempty we may be justified in eliminating noncore outcomes from further consideration.

Theorem 3. *Every quasibalanced NTU game has a nonempty core.*

This result was proved by Scarf (1967). The implication holds in one direction

only; in contrast to the TU case, one does not have equivalence. Moreover, because every balanced game is quasibalanced, Scarf's Theorem implies that every balanced game (as defined above) has a nonempty core.

Now let us return to our model of an exchange economy and show how it generates a well-behaved game with nontransferable utility. As before, we permit each coalition to redistribute its own resources provided that every coalition member receives an allocation belonging to the consumption set \mathbb{R}_+^ℓ . Accordingly, define $V : 2^N \rightarrow \mathbb{R}^n$ by $V(\emptyset) = \mathbb{R}^n$ and for each nonempty $S \subseteq N$, $V(S) = \{(w_1, \dots, w_n) \in \mathbb{R}^n \mid \text{there exists } (x_1, \dots, x_n) \in \mathbb{R}^{\ell n} \text{ with } \sum_{i \in S} x_i \leq \sum_{i \in S} e_i, x_i \in \mathbb{R}_+^\ell \text{ for each } i \in S, \text{ and } w_i \leq u_i(x_i) \text{ for all } i \in S\}$.

By definition, the $V(S)$ are comprehensive cylinder sets. They're compactly generated (and, hence, closed as the sum of a closed set and a compact set) by the upper semicontinuity of utility functions. This implies that each $V(S)$ set is bounded above in all of the coordinates corresponding to players in S or, equivalently, that the $V(S)_S$ sets are bounded above. Moreover, concavity of utility functions implies that each $V(S)$ or $V(S)_S$ set is convex. ✓

Finally, the economic model specified above gives rise to an NTU game which is totally balanced. The proof uses convex combinations of feasible allocations and concavity of utilities. This implies the following desirable property.

Theorem 4. *A finite pure exchange economy, having n agents $i = 1, \dots, n$ with consumption sets \mathbb{R}_+^ℓ , initial endowments $e_i \in \mathbb{R}_+^\ell$, and utilities $u_i : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$ which are assumed to be concave and upper semicontinuous, generates a totally balanced NTU game so that the game and all of its subgames have nonempty cores.*

Billera (1974), Billera and Bixby (1974), and Mas-Colell (1975) examine whether totally balanced NTU games satisfying the properties mentioned above can be generated by economies. The results are less sharp than those for the TU case and require technical restrictions which are not discussed here.

An extremely useful reference for much of this material is the book by Hildenbrand and Kirman (1976). The Shapley and Shubik (1969) article is also accessible. Needless to say, all students interested in cooperative game theory should read the following papers relating balancedness to the property of having a nonempty core: Bondareva (1962), Shapley (1967), and Scarf (1967).

3. Economies with Asymmetric Information

This section explains how one can add information to the basic model of a pure exchange economy. We are interested in situations in which different agents may initially possess different information. Moreover, the information must matter to traders.

To model these phenomena, we begin with an arbitrarily given abstract set Ω of states of the world. Elements ω of the set Ω are assumed to completely describe the relevant uncertainty in the universe. A σ -field \mathcal{F} of measurable subsets of Ω is also given. Subsets of Ω that belong to \mathcal{F} are also termed *events*. Technically, (Ω, \mathcal{F}) is a measurable space. Finally, (Ω, \mathcal{F}) is endowed with a (σ -additive) probability measure μ . [This could be generalized to permit agents to have different subjective probabilities regarding the *ex ante* likelihood of various events in Ω , provided that all agree about the null events—those which occur with probability zero.]

The information of trader $i \in N$ is given by a sub- σ -field \mathcal{G}_i of \mathcal{F} . Notice that information becomes an *ex ante* concept, in that it means the capacity to condition one's actions on a particular sub- σ -field, where the agent knows which sub- σ -field can be used. Thus, information is like an entire random variable (or measurable function from Ω to \mathbb{R}), rather than a single observation of the random variable (or a real number which equals the function evaluated at a specific $\bar{\omega} \in \Omega$). Another analogy is that one should think of information as access to an instrument or measuring device, not as a measurement which is the output of the instrument. In particular, information is not equivalent to the fact that a certain state $\bar{\omega}$ has actually occurred. Note that asymmetric information is sometimes called *differential information*, while *incomplete information* properly

refers to situations in which \mathcal{G}_i is smaller than \mathcal{F} , regardless of whether the \mathcal{G}_i may be different for different agents. Symmetric information is a special case of asymmetric information, and complete information is a special case of incomplete information.

A simpler model which captures most of the main ideas starts from a finite set Ω of states of the world, where each state occurs with strictly positive probability. Agents' information is specified by partitions of Ω . When state ω occurs, the agent learns the (unique) element of the partition containing ω .

States of the world can also be interpreted as signals (about some underlying fundamental states of the world). However, rather than using dual terminology to include this case, I prefer to think of a state of the world as an n -tuple of the signals that have been received by each agent.

The state of the world can affect traders' endowments and utilities. We formalize this by two measurable functions defined on Ω . For each $i \in N$, trader i 's initial endowments are given by $e_i : \Omega \rightarrow \mathbb{R}_+^\ell$ which is \mathcal{G}_i -measurable. The restriction to \mathcal{G}_i -measurability (rather than \mathcal{F} -measurability) means that trader i must know his or her own initial endowment; the endowment vector can depend only on the trader's own information. If Ω is infinite, we assume further that each e_i is uniformly bounded almost surely in order to avoid technicalities; this condition is automatically satisfied for finite Ω . State-dependent utilities are frequently written as functions $u_i : \mathbb{R}_+^\ell \times \Omega \rightarrow \mathbb{R}$ which are continuous on \mathbb{R}_+^ℓ and \mathcal{F} -measurable on Ω (so that they're jointly measurable). We use \mathcal{F} -measurability instead of \mathcal{G}_i -measurability, because we envision that traders eventually learn their true utilities upon consumption of their allocations, but they may not fully know their state-dependent utilities when they make trades or choose strategies. The essential uncertainty here pertains to one's own preferences. We assume (again, to avoid potential difficulties of a technical nature) that for almost all $\omega \in \Omega$ and all $i \in N$, the utility functions $u_i(\cdot; \omega) : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$ are not only continuous, but also strictly concave and strictly monotone. [For technical reasons (based on the fact that proper regular

conditional probability distributions are defined only up to null sets) state-dependent utilities should be specified by \mathcal{F} -measurable functions $U_i : \Omega \rightarrow C(\mathbb{R}_+^\ell, \mathbb{R})$, where the space $C(\mathbb{R}_+^\ell, \mathbb{R})$ of continuous functions from \mathbb{R}_+^ℓ to \mathbb{R} is endowed with the Borel σ -field corresponding to the topology of uniform convergence on compact subsets, which makes $C(\mathbb{R}_+^\ell, \mathbb{R})$ into a Frechet space. All conditional expectations are taken with respect to the induced image measure on $C(\mathbb{R}_+^\ell, \mathbb{R})$, not on the abstract probability space $(\Omega, \mathcal{F}, \mu)$. Here one assumes that for all $i \in N$ and for almost all $\omega \in \Omega$, $U_i(\omega)$ is strictly monotone and strictly concave; it may also be convenient or necessary to assume that the (unconditional) distribution on $C(\mathbb{R}_+^\ell, \mathbb{R})$ has compact support for all $i \in N$.]

An important conceptual problem with asymmetric information models is that one must carefully delineate those actions (i.e., trades or strategies) among which agents may choose. Radner (1968) considers the question of what people can do in a market when they have asymmetric information. He proposes that one should be able to verify one's own (net) trades. For example, you will never pay a strictly positive amount to sign a contract with me stating that I will give you \$100 if I do not have a headache tomorrow morning. If you do, I can always tell you that I have a headache, and you can never know that I'm lying. [You also can't prove to a third party that I'm lying, which exemplifies the issue of verification rather than asymmetric information.] In competitive equilibrium models, agents trade impersonally with the market, which means that one's own net trade should depend only on information available to the agent at the time the market meets. Hence, the individual excess demand of agent i should be \mathcal{G}_i -measurable, which implies (because e_i is \mathcal{G}_i -measurable) that i 's allocation is also \mathcal{G}_i -measurable. Radner (1968) demonstrates that, in such models in which consumers have different consumption sets because of the restrictions to subspaces of \mathcal{G}_i -measurable functions from Ω to \mathbb{R}_+^ℓ , competitive equilibria exist, provided that Ω or all of the \mathcal{G}_i are finite.

However, the appropriate informational restrictions in cooperative games are less clear-cut. Do agents share their information freely within a coalition, or can coalitions

only make binding agreements based on information which is common to all members? The same problem arises when one attempts to define Pareto optimality in asymmetric information models. [A different approach involving interim efficiency is explored by Holmstrom and Myerson (1983) and more recently by Forges (1990, 1991).]

4. Wilson's Article

In a seminal article, Wilson (1978) examines the core of an economy with asymmetric information. He focuses on the need to define the information of players in a coalition when they (initially) have access to different information.

The analysis is performed in a pure exchange environment with finitely many states. Initial endowments are assumed to be always measurable for every agent in every coalition. Wilson (1978) first defines the abstract concept of *communication structures* and then focuses on two special extreme cases: the *coarse core*, defined by the condition that the information for all players in coalition S is precisely the sub- σ -field $\bigwedge_{i \in S} \mathcal{G}_i$ of information that they have in common, and the *fine core*, defined by giving every member of coalition S the sub- σ -field $\bigvee_{i \in S} \mathcal{G}_i$ of pooled information, so that the coalition can use any information that was initially available to any of its members.

Wilson (1978) then examines whether these two cores are nonempty. The intuition is as follows: The use of only common information renders blocking difficult, so that the coarse core is expected to be nonempty. However, blocking is easy with pooled information, so that the fine core may be empty. To prove nonemptiness of the coarse core, Wilson (1978) argues that a game he defines, in which each "player" consists of a state-player pair, is balanced. For the fine core, he provides a counterexample with three states and three players in which any feasible, efficient allocation for the grand coalition can be blocked in some state by some coalition.

However, when one contemplates these results in light of the market games literature and the relationships between balanced games and those with nonempty cores, the intuition about easy versus difficult blocking seems problematic. Potential blocking al-

locations should be compared to the set of allocations available to the grand coalition. As in the earlier market games literature, one examines convex combinations (with balancing weights) of feasible allocations. Concavity guarantees that the utilities of convex combinations dominate the average utilities of original allocations.

For the coarse core, this argument seems to fail. Convex combinations of $\bigwedge_{i \in S} \mathcal{G}_i$ -measurable and $\bigwedge_{i \in T} \mathcal{G}_i$ -measurable functions need not be $\bigwedge_{i \in S \cup T} \mathcal{G}_i$ -measurable. However, the paradox is solved when one notices that Wilson's (1978) model uses $\bigwedge_{i \in S} \mathcal{G}_i$ as the information for subcoalitions $S \subset N$ with $S \neq N$ and reverses this logic to take $\bigvee_{i \in N} \mathcal{G}_i$ as the information for the grand coalition N . This explains the apparent inconsistency of the two strategies for proving that the core is nonempty.

In contrast, the argument that market games are balanced seems to apply to the fine core, as all convex combinations are measurable with respect to the information $\bigvee_{i \in N} \mathcal{G}_i$ of the grand coalition. Yet, detailed examination of Wilson's (1978) counterexample indicates that the blocking he employs to show that the core is empty must occur *ex post*. In Wilson's (1978) argument, some coalition blocks a given feasible allocation by dominating it in some particular state of the world. This would be consistent with a parallel state-by-state definition of the feasible and efficient allocations, so that in this case the economy with asymmetric information essentially reduces to three distinct economies in which all agreements and all trades take place after agents learn their information about the particular state of the world that has occurred.

Kobayashi (1980) obtains some results extending Wilson's (1978) coarse core using the concept of common knowledge. He also permits the set Ω of states of the world to be infinite.

5. Market Games with Asymmetric Information

In order to study cooperative solution concepts for economies with asymmetric information more systematically, one must derive the TU or NTU games that are generated by such economies. Standard results from game theory then apply, provided that

the induced games are well defined and satisfy the necessary assumptions. Moreover, failures of certain solution concepts—such as the potential emptiness of the core—can be understood in terms of the game theoretic hypotheses that are violated as a consequence of asymmetric information. My formulation of market games with asymmetric information is based on *ex ante* agreements within coalitions. In particular, blocking can occur only before agents learn about the state of the world that has occurred, so that all payoffs in the resulting games consist of (unconditional) expected utilities (of information-conditional allocation functions). An advantage of this approach is that it enables agents to engage in risk-sharing trades and to write contracts that Pareto dominate those available with *ex post* agreements and *ex post* blocking.

Before the market games can be defined, one must specify the information available to every agent in every coalition. Let \mathcal{H}_i^S denote the information that agent $i \in S$ can use as a member of coalition S . Assume that \mathcal{H}_i^S is a sub- σ -field of \mathcal{F} and that $e_i : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$ is measurable with respect to \mathcal{H}_i^S for all $S \ni i$ and all $i \in N$. Note that all members of a coalition need not be restricted to the same information; $\mathcal{H}_i^S \neq \mathcal{H}_j^S$ is permitted. A natural assumption is that $\mathcal{H}_i^S = \mathcal{G}_i$ for all $S = \{i\}$ and all $i \in N$.

Given an economy with asymmetric information as modeled in Section 3 and given the sub- σ -fields \mathcal{H}_i^S for all $S \subseteq N$ ($S \neq \emptyset$) and all $i \in S$, define the induced cooperative game $v : 2^N \rightarrow \mathbb{R}$ with transferable utility by $v(\emptyset) = 0$ and $v(S) = \max \left\{ \sum_{i \in S} \int u_i(x_i(\omega); \omega) d\mu(\omega) \mid \text{for all } i \in S, x_i : \Omega \rightarrow \mathbb{R}_+^\ell \text{ is } \mathcal{H}_i^S\text{-measurable and } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega) \text{ a.s.} \right\}$ for all $S \subseteq N$ with $S \neq \emptyset$.

Theorem 5. *The induced TU game v defined above is well defined.*

To show that the game is well defined requires proving that the maximum exists. Doing so gives rise to technical difficulties (to be discussed briefly below) whenever Ω is finite.

Similarly, one can define the derived *NTU* games. Let $V : 2^N \rightarrow \mathbb{R}^n$ be defined by $V(\emptyset) = \mathbb{R}^n$ and, for $S \subseteq N$ with $S \neq \emptyset$, $V(S) = \{(w_1, \dots, w_n) \in \mathbb{R}^n \mid \text{there exist } \mathcal{H}_i^S\text{-}$

measurable functions $x_i : \Omega \rightarrow \mathbb{R}_+^\ell$ with $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$ a.s. such that $w_i \leq \int_{\Omega} u_i(x_i(\omega); \omega) d\mu(\omega)$ for all $i \in S$ }.

Theorem 6. *The induced NTU game is well defined. Moreover, for all $S \subseteq N$, the $V(S)$ sets are convex, and they are compactly generated whenever $S \neq \emptyset$.*

Closedness of the $V(S)$ sets is roughly equivalent to existence of the maxima for TU games; this is difficult when Ω fails to be finite. Convexity follows from concavity of utilities, while the property of being compactly generated comes from the uniform boundedness of initial endowments and upper semicontinuity of utilities.

If Ω is infinite, the argument exploits the characterization of weakly and strongly compact convex subsets in \mathcal{L}^1 spaces, especially the theorem of Dunford and Pettis (1940). [See Dunford and Schwartz (1958) or Rudin (1973) for technical background material.] Details appear in Allen (1991a, 1991b, 1991c). Page (1993) extends these theorems to allow the underlying commodity space \mathbb{R}^ℓ to be replaced by an infinite-dimensional space.

6. Cores with Asymmetric Information

In this section, we utilize balancedness conditions on the derived TU and NTU games to obtain nonempty cores with asymmetric information. A summary of the finite state case appears in Allen (1994), while the basic general references are Allen (1991b, 1991c).

Theorem 7. *A sufficient condition for balancedness of the derived TU or NTU game is that for all coalitions $S \subseteq N$ and all agents $i \in S$, $\mathcal{H}_i^S \subseteq \mathcal{H}_i^N$. Total balancedness holds if $\mathcal{H}_i^S \subseteq \mathcal{H}_i^T$ whenever $i \in S \subseteq T \subseteq N$. In particular, if $\mathcal{H}_i^S \subseteq \mathcal{H}_i^N$ whenever $i \in S$, then the core is nonempty, while $\mathcal{H}_i^S \subseteq \mathcal{H}_i^T$ whenever $i \in S \subseteq T \subseteq N$ implies that the cores of all submarkets are nonempty.*

The proof is based on the observation that sums of functions measurable with respect to different sub- σ -fields are measurable with respect to the smallest σ -field generated by all of the sub- σ -fields. The second statement follows from Bondareva (1962) and

Shapley (1967) or Scarf (1967).

Theorem 7 implies that the fine information [i.e., $\mathcal{H}_i^S = \sigma\left(\bigcup_{j \in S} \mathcal{G}_j\right)$ whenever $i \in S$] core in the sense defined here with *ex ante* blocking is nonempty. It also implies that the private information [i.e., $\mathcal{H}_i^S = \mathcal{G}_i$ for all $i \in S$ and all coalitions $S \subseteq N$] core is nonempty. It does not apply to the coarse information [$\mathcal{H}_i^S = \bigcap_{j \in S} \mathcal{G}_j$ for $i \in S \subseteq N$] core and, in fact, counterexamples are not too difficult to find. However, a consequence of the theorem is that Wilson's coarse core [$\mathcal{H}_i^S = \bigcap_{j \in S} \mathcal{G}_j$ for $i \in S$ if $S \neq N$ and $\mathcal{H}_i^N = \sigma\left(\bigcup_{j \in N} \mathcal{G}_j\right)$] is necessarily nonempty. Of course, many other specifications for information sharing within coalitions are possible, and the theorem provides sufficient (but not necessary) conditions for such models to yield nonempty cores.

Yannelis (1991) shows that exchange economies have private information core allocations. Allen (1992) provides a different proof, under somewhat different assumptions, which follows directly from the market games approach. In Allen (1993a), private information sharing is related to a condition, termed *publicly predictable information*, stating that any single agent's information can always be deduced from the pooled information of all other coalition members.

7. Values with Asymmetric Information

Having derived cooperative games from economies with asymmetric information, one can apply any of the myriad of alternative (*TU* or *NTU*) solution concepts, provided that the requisite hypotheses are satisfied by the derived game. The value is only one of many solution concepts, albeit it is one that has nice properties and has proved to be extremely useful for many problems in economics. Thus, it is discussed here for illustrative purposes.

Theorem 8. *The TU Shapley value of the derived game exists and is unique. The NTU value exists if $\mathcal{H}_i^S \subseteq \mathcal{H}_i^N$ whenever $i \in S$.*

The qualification for *NTU* games is needed for monotonicity of the λ -transfer games. See Allen (1991a) for details, which use results from Aumann and Shapley (1974,

Appendix A) and Shapley (1969).

Krasa and Yannelis (1994) show via a direct argument that economies with asymmetric information have value allocations. They do not examine all of the information sharing possibilities that are covered by the above argument.

8. The Harsanyi Approach

A different framework for the analysis of information in cooperative game theory is based on Harsanyi's (1967-68) formalization of noncooperative games with incomplete information. Recall that a noncooperative game is specified by a player set $N = \{1, \dots, n\}$, strategy sets \mathcal{S}_i for each $i \in N$, and payoff functions $U_i : \prod_{i \in N} \mathcal{S}_i \rightarrow \mathbb{R}$ for each $i \in N$. To capture the notion of incomplete information, Harsanyi (1967-68) replaces the single payoff function for each player by payoff functions that are parameterized by a type space. The basic idea is that a *type* for player i is taken to consist of the player's own payoff function and his or her beliefs about the payoff functions of other players, which are distributions (possibly depending on the player's own type) over the product of other players' type spaces.

If one contemplates this approach in the context of cooperative theory, problems arise even with transferable utility. For instance, when i learns his or her own type with certainty, does player i then know the entire game $v : 2^N \rightarrow \mathbb{R}$ in characteristic function form—in which case, either beliefs are inconsistent or there is no asymmetric information—or does player i only have some belief about the correct distribution over possible characteristic functions? How do players make enforceable agreements within coalitions when they believe they're playing different games—i.e., when their beliefs over $v(S)$ are different? Even if coalition members agree about v , they may disagree about how they can actually achieve the maximal worth of their coalition. For example, you and your spouse could both believe that you can double your household wealth, but you may disagree over whether this can be done by buying orange juice futures or by shorting Singapore stocks. What, then, is the worth of such a coalition?

These problems can be interpreted as suggesting the need to include functions from strategy sets or actions to payoffs as part of the primitive description of a cooperative game with incomplete information. Under complete or symmetric information, cooperative game theory usually suppresses any explicit notion of strategies or actions, although implicitly, when we say that $v(S)$ is the worth of coalition S , we really mean that the players of coalition S together have some (feasible) joint strategy that enables them to earn $v(S)$. Under incomplete information, the strategies should be explicitly included in our cooperative games. Observe that this issue did not arise for market games with asymmetric information because we did write the strategies in the definition of $v(S)$ and $V(S)$; the strategies were state-dependent net trades (or state-dependent allocations) for each player in the coalition.

A pathbreaking article by Myerson (1984) explores such a formulation of cooperative games with incomplete information. For each coalition S , let \mathcal{D}_S denote the set of actions available to S , and assume that $\mathcal{D}_S \times \mathcal{D}_T \subseteq \mathcal{D}_{S \cup T}$ when $S \cap T = \emptyset$. [This is a superadditivity condition.] Take the set \mathcal{T}_i of types for player i to be a finite set for all $i \in N$; assume that all combinations of types [i.e., all n -tuples $(t_1, \dots, t_n) \in \prod_{i \in N} \mathcal{T}_i$] occur with strictly positive probability. Write $\mathcal{T}_S = \prod_{j \in S} \mathcal{T}_j$ for the set of profiles of types in the coalition S . Let $\mathbf{u}_i(d, t_1, \dots, t_n)$ be the payoff to player $i \in N$ if the grand coalition N chooses strategy $d \in \mathcal{D}_N$ when players' types are $(t_1, \dots, t_n) \in \mathcal{T}$. This model permits externalities, although there is no obvious way to define subgames except by having the subgames depend on some given type realization and action of the complementary coalition. [This situation is worse than in cooperative games with complete information in that, while a coalition can perhaps observe the action of its complement, the coalition may have no way to ascertain the type drawings of players who do not belong to the coalition.] Myerson (1984) further assumes the consistency condition of Harsanyi (1967-68) that there exists a probability p on \mathcal{T} such that its conditional distributions satisfy $p_i(t_{-i}|t_i) = p(t) / \sum p(t_i, s_{-i})$, where the summation is taken over

$s_i \in \mathcal{T}_i$. Then a *cooperative game with incomplete information* and player set N is defined by $((\mathcal{D}_S)_{S \subseteq N, S \neq \emptyset}, (\mathcal{T}_i, \mathbf{u}_i)_{i \in N}, p)$ satisfying the above assumptions. Myerson (1984) studies bargaining solutions in such games.

This model forms the basis for recent research by Allen (1993b), Ichiishi and Idzik (1992), and Rosenmüller (1990), among others. As this work focuses on issues of incentive compatibility and, hence, relates to implementation, I do not discuss it further here.

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