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THE PERMANENT INCOME HYPOTHESIS REVISITED

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ABSTRACT

Measured aggregate U.S. consumption does not behave like a martingale. This paper develops and tests two variants of the permanent income model that are consistent with this fact. In both variants, we assume agents make decisions on a continuous time basis. According to the first variant, the martingale hypothesis holds in continuous time and serial persistence in measured consumption reflects only the effects of time aggregation. We investigate this variant using both structural and atheoretical econometric models. The evidence against these models is far from overwhelming. This suggests that the martingale hypothesis may yet be a useful way to conceptualize the relationship between aggregate quarterly U.S. consumption and income. According to the second variant of the permanent income model, serial persistence in measured consumption reflects the effects of exogenous technology shocks and time aggregation. In this model, continuous time consumption does not behave like a martingale. We find little evidence against this variant of the permanent income model. It is difficult, on the basis of aggregate quarterly U.S. data, to convincingly distinguish between the different continuous time models considered in the paper.

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1. Introduction

Few subjects in macroeconomics have received as much attention as the relationship between aggregate consumption and aggregate income. This attention reflects, at least in part, the belief that an understanding of the structural determinants of aggregate consumption is central to resolving many of the issues in business cycle theory. During the past decade, much of the empirical literature on aggregate consumption has centered on Hall's (1978) demonstration that, under certain conditions, the permanent income hypothesis (PIH) implies that consumption ought to be a martingale. Under the martingale hypothesis (MH), no variable other than current consumption should help predict future consumption.

In fact, a number of authors, including Flavin (1981) and Hayashi (1982), have reported statistically significant correlations between the change in consumption and lagged consumption and income. The response to these findings has generally fallen into one of two categories. First, some researchers have attributed this "excess sensitivity" of consumption to the presence of a substantial number of consumers who are liquidity constrained. Under this view, the PIH is fundamentally flawed as a principle for organizing the aggregate time series data. [See, for example, Hall and Mishkin (1982) and Zeldes (1989).]

A second view of the empirical shortcomings of the MH is that they do not reflect the failure of the PIH per se. Instead, they reflect the failure of the auxiliary assumptions required to derive the MH from the PIH. This view underlies both intertemporal capital asset pricing models [for example, Hansen and Singleton (1982, 1983), Dunn and Singleton (1986), Eichenbaum and Hansen (1989)] and real business cycle theories [for example, Kydland and Prescott (1982), Long and Plosser (1983), Michener (1984)] that abstract from liquidity constraints and other market imperfections which would prevent consumers from optimally adjusting consumption to permanent income.

This paper pursues the second of these two views. We investigate whether two perturbations of the versions of the PIH implemented by Hall (1978) and Flavin (1981) are consistent with the quarterly aggregate U.S. consumption and output data. First, we replace the assumption that agents' decision intervals match the data sampling interval with the assumption that agents make decisions at time intervals finer than that. Second, we change the technology for producing consumption goods to one that no longer implies the MH.

Our first modification is motivated by the results of Sims (1971), Geweke (1978), Christiano (1984, 1985), and Marcet (1986), among others, which indicate that temporal aggregation bias can induce spurious serial correlation and spurious Granger–causality findings. In fact, much of the empirical evidence against different versions of the PIH consists of findings that the first difference of aggregate consumption is serially correlated and is Granger–caused by a variety of other variables. If agents make economic decisions at intervals of time finer than the data sampling interval, then these serial correlation and Granger–causality findings could be spurious in the sense that they reflect only the effects of temporal aggregation bias.

We explore the implications of temporal aggregation bias for tests of the MH in Section 2. In order to do this, we formulate and test continuous time analogs of the discrete time MHs considered by Hall (1978) and Flavin (1981). We find that, in sharp contrast to the discrete time MH, our data set has surprisingly little evidence against the continuous time MH. In our view, the evidence against the continuous time MH is sufficiently weak to warrant the investigation of continuous time general equilibrium models which embody this hypothesis. At the same time, our reservations about the continuous time MH are sufficiently strong to warrant the investigation of models which incorporate the PIH but do not imply that continuous time consumption is a martingale.

In Section 3 of the paper, we describe a continuous time stochastic growth model in which output is produced using both labor and capital according to a Leontief–type production function. As do Hansen (1987) and Sargent (1987), we interpret the PIH as an implication of a simple version of the Brock–Mirman growth model. When the labor requirement per unit of capital is nonstochastic and the subjective rate of time preference equals the net marginal product of capital, our model implies that consumption obeys the MH. When either of these conditions does not hold, however, consumption does not act like a martingale.

In Section 4, we empirically implement different versions of this growth model using techniques developed by Hansen and Sargent (1980) for estimating continuous time models from discrete time data. This strategy allows us to directly address the possibility of temporal aggregation bias and to explicitly account for the fact that consumption and income data are not point–in–time sampled. Our main finding is that none of the versions of the model which we investigate is decisively rejected by the data. We also find, however, that certain versions of the model which do not imply the continuous time MH do perform marginally better than the version of the model which embodies the continuous time MH.

In Section 5, we make some concluding remarks.

2. Evidence on the Discrete and Continuous Time Martingale Hypotheses

Above we suggested that the empirical shortcomings of the discrete time MH could, in principle, be due to temporal aggregation bias. To test this conjecture, we examine the empirical plausibility of the martingale version of the PIH under two alternative assumptions regarding the timing of agents' decisions. First, we assume that the decision intervals of private agents coincide with the data sampling interval, and we investigate whether quarterly consumption acts like a martingale. We refer to this as the <u>discrete time martingale hypothesis</u>. Second, we suppose that agents' decision intervals are finer than the data sampling interval. For simplicity, we consider the limiting case in which agents make decisions on a continuous time basis. Then the hypothesis of interest is whether continuous time consumption acts like a martingale. We refer to this as the <u>continuous time martingale hypothesis</u>. Both the discrete and continuous time MHs imply restrictions on the time series properties of consumption which can be tested using the generalized method of moments (GMM) procedures developed by Hansen (1982).

Consistent with previous studies, we find that the discrete time MH is strongly rejected by the consumption data. The continuous time MH provides a better characterization of the data, in the sense that our tests fail to reject it at conventional significance levels.

2.A. Tests of the Discrete Time MH

We begin with a formal definition of the discrete time MH.

Definition: The random variable x_t is said to satisfy the discrete time martingale hypothesis (DTMH) if the level of x_t is a martingale with deterministic, possibly time–varying drift; that is,

$$(2.1) \qquad E_t x_{t+\tau} = x_t + f_{t,\tau},$$

where $f_{t,\tau}$ is deterministic and is defined for integer values of t and for $\tau = 1, 2, ...$

Throughout, E_t denotes the time t conditional expectations operator. We assume that x_t is contained in the time t information set.

We include the drift term, $f_{t,\tau}$, to accommodate the fact that per capita U.S. consumption and output have displayed persistent growth over time. The structural model we present in Section 3 implies that consumption, output, and the first difference of consumption are rendered stationary when detrended by the common geometric growth rate ϕ. We impose these growth restrictions as an assumption and set

$$(2.2) \qquad f_{t,\tau} = \phi^{t+1}[(1-\phi^{\tau})/(1-\phi)]C^{o},$$

where C^{o} is a positive constant. Let c_t denote time t consumption. Given (2.2), the DTMH applied to c_t implies that

$$E_t \phi^{-(t+1)} \Delta c_{t+1} = C^o, \tag{2.3}$$

where Δ denotes the first–difference operator.

Restriction (2.3) can be tested using the procedures discussed by Hansen (1982). Define the function

$$\psi(C, \phi^{-t}\Delta c_t) = \phi^{-t}\Delta c_t - C. \tag{2.4}$$

To simplify notation, we write

$$\psi_t(C) = \psi(C, \phi^{-t}\Delta c_t). \tag{2.5}$$

Relation (2.3) implies that $E_{t-1}\psi_t(C^o) = 0$, so that $E\psi_t(C^o)z_{it} = 0$ for all instruments z_{it} contained in consumers' common time $t-1$ information sets, I_{t-1}. In our empirical work, we consider the instrument vectors

$$\begin{aligned} z_{1t} &= [1, \phi^{-(t-1)}\Delta c_{t-1}, \ldots, \phi^{-(t-4)}\Delta c_{t-4}, \phi^{-(t-1)}\tilde{y}_{t-1}, \ldots, \phi^{-(t-4)}\tilde{y}_{t-4}]', \\ z_{2t} &= [1, \phi^{-(t-1)}\Delta c_{t-1}, \ldots, \phi^{-(t-4)}\Delta c_{t-4}, \\ &\qquad \phi^{-(t-1)}(c_{t-1}-\tilde{y}_{t-1}), \ldots, \phi^{-(t-4)}(c_{t-4}-\tilde{y}_{t-4})]', \end{aligned} \tag{2.6}$$

where $\tilde{y}_t$ denotes gross output at time t. Under the assumption that $\phi^{-t}\Delta c_t$ and z_{it} are jointly stationary and ergodic, the GMM procedure described by Hansen (1982) can be used to estimate the parameter C^o and test the null hypotheses, $E\psi_t(C^o)z_{it} = 0$ for $i = 1, 2$.

To describe this procedure, we define the functions $g_{iT}(C) = (1/T)\Sigma_{t=1}^{T}\psi_t(C)z_{it}$ for $i = 1, 2$. Our estimator of C^o, $\hat{C}_i$, is the argmin of $g_{iT}'(C)W_{iT}^{-1}g_{iT}(C)$ for $i = 1, 2$. Here W_{iT} is the sample covariance of $\psi_t(\tilde{C}_i)z_{it}$, where $\tilde{C}_i$ is the argmin of $g_{iT}'(C)g_{iT}(C)$ for

$i = 1, 2$. Then $J_{iT} = g_{iT}'(\hat{C}_i)W_{iT}^{-1}g_{iT}(\hat{C}_i)$ for $i = 1, 2$ is asymptotically distributed as a chi–square random variable with eight degrees of freedom. We use these statistics to test relation (2.3).

In implementing this procedure, we used the following data set. Initially, we measured total consumption, c, as the sum of total government consumption, c_g, and private consumption. The latter was measured as real expenditures on nondurable consumption goods (c_{nd}) and services (c_s) plus an estimate of the service flow from the stock of consumer durables (c_{sd}). All of these measures, except c_{sd} and c_g, were taken from the U.S. national income and product accounts (NIPA). We obtained an estimate of c_{sd} from the data base documented by Brayton and Mauskopf (1985). We measured c_g by NIPA real government purchases of goods and services (g) minus real government investment (i_g). A measure of i_g was provided to us by John Musgrave of the U.S. Bureau of Economic Analysis.[1] Gross output, $\bar{y}$, was measured as real gross national product (GNP) plus c_{sd}. All series are quarterly, cover the period 1950.2–1985.3, and are converted to per capita terms using a measure of total population that includes armed forces overseas.[2] The value of ϕ used to construct the detrended data is $\exp(\theta)$, where $\theta = 0.004568$, the coefficient on time of the regression of $\log(c_t)$ and $\log(\bar{y}_t)$ on a linear time trend and a constant, computed subject to the restriction that the growth rates of consumption and output are equal.[3] We treat ϕ as a known constant, so sampling error in estimating ϕ is ignored in the hypothesis tests reported below.

Tests of (2.3) based on c and $\bar{y}$ are reported in Table I. (The entries there contain the significance levels of our test statistics.) Table I reports results based on the instrument vectors z_{1t} and z_{2t}. As a check on robustness, we report calculations based on three sample periods. For all three periods and for both instrument vectors, the DTMH is strongly rejected. As a further check on robustness, we also report results for alternative measures of consumption and income used by other researchers. The columns marked "c_{nd}, y_d" report results for the consumption and income concepts used by Flavin (1981). Here y_d denotes per capita disposable income. The columns marked "$c_{nd} + c_s$, y_d" report results for the consumption and income concepts used by Hall (1978). Following Flavin, we detrended these measures of consumption and income using different, though constant, geometric growth rates.[4] The results using these alternative data series confirm our strong evidence against the DTMH since, with only two exceptions, we can reject this hypothesis at the 5 per cent significance level or higher.

2.B. Tests of the Continuous Time MH

We now abandon the assumption that agents' decision intervals coincide with the data sampling interval. Instead, we assume that agents make decisions on a continuous time basis. Throughout, we follow the convention of placing time indices of continuous time random variables in parentheses.

We define the continuous time MH as follows.

Definition: The random variable x(t) is said to satisfy the continuous time martingale hypothesis (CTMH) if and only if

$$E_t x(t+\tau) = x(t) + f(t, \tau), \tag{2.7}$$

where $\tau > 0$ and $f(t, \tau)$ is a deterministic, possibly trivial, function of t. We assume that x(t) is contained in the time t information set.

The continuous time analog to (2.2) is

$$f(t, \tau) = \phi^{t+1}[(1-\phi^{\tau})/(1-\phi)]C^{o}, \tag{2.8}$$

for integer values of t and for $\tau > 0$.

If c(t) satisfies the CTMH, then measured quarterly consumption will not obey the DTMH. From the perspective of the continuous time model, measured consumption is the time average of consumption over the discrete time sampling interval, which in our case equals one quarter. It follows that measured Δc_t is a weighted integral of innovations to the underlying continuous time consumption process from the beginning of quarter $t-1$ to the end of quarter t. Since the quarter $t-1$ innovations also appear in Δc_{t-1}, the CTMH implies that Δc_t and Δc_{t-1} have nonzero covariance. In addition, Δc_{t-1} and y_{t-1} ought to be correlated because continuous time output will be correlated with previous innovations to continuous time consumption and y_{t-1} is the average value of output during quarter $t-1$. For these reasons, temporal aggregation can, in principle, account for the rejection of the DTMH discussed above. Consequently, we test the orthogonality conditions implied by the CTMH as well its implications for the autocorrelation structure of $\phi^{-t}\Delta c_t$. In this context, we exploit a slight generalization of results by Working (1960).

Suppose that consumption does not grow, so that $\phi = 1$ and $f(t, \tau) = \tau C^o$ in (2.8). Also, let $\bar{x}(t)$ denote the time average of $x(t)$, $\bar{x}(t) = \int_0^1 x(t+\tau)\, d\tau$. Working (1960) established that, under these circumstances, the first difference of the time–averaged random walk, $\Delta\bar{x}(t)$, has an autocorrelation coefficient of 0.25 at lag 1 and zero elsewhere. Christiano and Marshall (1987) showed that, after rounding to three digits, this result is also valid for $\phi^{-t}\Delta\bar{x}(t)$ when $\phi = \exp(0.004568)$. Thus, the CTMH implies that $\psi_t(C^o)$, defined in (2.5), has mean zero and autocorrelation coefficients of $\rho_1^o = 0.25$ and $\rho_2^o = 0$ at lags 1 and 2, respectively. Here $\rho_i^o = E[\psi_t(C^o)\psi_{t-i}(C^o)]/E[\psi_t(C^o)]^2$ for $i = 1, 2$. Define the function

$$(2.9) \qquad H_t(\rho_1, C) = [\psi_t(C),\ \psi_t(C)\psi_{t-1}(C) - \rho_1\psi_t(C)^2]'.$$

Under the CTMH, $E_{t-2}H_t(\rho_1^o, C^o) = 0$, which implies the set of unconditional moment restrictions $EH_t(\rho_1^o, C^o) = 0$. These can be exploited to estimate the parameters C^o and ρ_1^o using Hansen's (1982) GMM procedures. Define the function $g_T(\rho_1, C) = (1/T)\Sigma_{t=1}^T H_t(\rho_1, C)$. Our estimator of (ρ_1^o, C^o), $(\hat{\rho}_1, \hat{C})$, is defined uniquely by the condition $g_T(\hat{\rho}_1, \hat{C}) = 0$. Let W_T be a sample estimate of the spectral density matrix of

$H_t(\hat{\rho}_1, \hat{C})$ evaluated at frequency zero.[5] Then our estimate of the variance–covariance matrix of $(\hat{\rho}_1, \hat{C})$ is $D_T^{-1}W_T(D_T')^{-1}$, where $D_T = [\partial g_T/\partial(\rho_1, C)]$ evaluated at $(\hat{\rho}_1, \hat{C})$.

Table II reports our point estimates and standard errors for ρ_1. Notice that for all measures of consumption and all sample periods, the estimated value of ρ_1 is well within one standard error of 0.25. We also used Hansen's (1982) procedure to estimate ρ_2. In no case can we reject, at the 5 per cent significance level, the null hypothesis that $\rho_2^o = 0$. Consequently, this set of tests provides virtually no evidence against the CTMH.

Next we tested the implication of the CTMH that $\phi^{-t}\Delta c_t$ is uncorrelated with elements of agents' time t − 2 information sets; that is, $E_{t-2}\psi_t(C^o) = 0$. This conditional moment restriction implies the set of unconditional moment restrictions $E\psi_t(C^o)\hat{z}_{it} = 0$ for all instruments $\hat{z}_{it}$ contained in I_{t-2}. In practice, $\hat{z}_{1t}$ and $\hat{z}_{2t}$ were specified as

$$(2.10)\qquad \begin{aligned} \hat{z}_{1t} &= [1, \phi^{-(t-2)}\Delta c_{t-2}, \ldots, \phi^{-(t-4)}\Delta c_{t-4}, \phi^{-(t-2)}\tilde{y}_{t-2}, \ldots, \phi^{-(t-4)}\tilde{y}_{t-4}]', \\ \hat{z}_{2t} &= [1, \phi^{-(t-2)}\Delta c_{t-2}, \ldots, \phi^{-(t-4)}\Delta c_{t-4}, \\ &\qquad \phi^{-(t-2)}(c_{t-2}-\tilde{y}_{t-2}), \ldots, \phi^{-(t-4)}(c_{t-4}-\tilde{y}_{t-4})]'. \end{aligned}$$

The null hypotheses $E\psi_t(C^o)\hat{z}_{it} = 0$ for $i = 1, 2$ were tested using the GMM procedure described above. Under our null hypotheses, the J_T statistics are asymptotically distributed as chi–square random variables with six degrees of freedom.

Significance levels of the computed test statistics appear in the "Lags 2–4" columns of Table I. Three features of these results are worth noting. First, in only one case do the significance levels in the "Lags 2–4" columns fail to exceed their counterparts in the "Lags 1–4" columns. Second, in only one case can we reject the null hypothesis that $E\psi_t(C^o)\hat{z}_{1t} = 0$ at the 1 per cent significance level. Third, in only one case can we reject the null hypothesis that $E\psi_t(C^o)\hat{z}_{2t} = 0$ at the 5 per cent significance level. Overall, then, this set of tests provides only weak evidence against the CTMH.

We conclude this section by reporting the results of testing a set of joint hypotheses: $\{E\psi_t(C^o)\hat{z}_{it} = 0,\ \rho_1^o = 0.25\}$ for i = 1, 2. Each of these null hypotheses implies the set of eight unconditional moment restrictions $EZ_{it}H_t(\rho_1^o, C^o) = 0$, where Z_{it} is the 8×2 block diagonal matrix with first and second diagonal blocks given by $\hat{z}_{it}$ and 1, respectively. Our estimator of the parameters ρ_1^o and C^o was the argmin of $g_{iT}'(\rho_1, C)W_{iT}^{-1}g_{iT}(\rho_1, C)$, where $g_{iT}(\rho_1, C) = (1/T)\Sigma_{t=1}^{T}Z_{it}H_t(\rho_1, C)$ and W_{iT} is a consistent estimate of the spectral density of $Z_{it}H_t(\rho_1^o, C^o)$ at frequency zero for i = 1, 2. In particular, W_{iT} is a sample estimate of the spectral density at frequency zero of $Z_{it}H_t(\tilde{\rho}_1, \tilde{C})$, where $\tilde{\rho}_1$, $\tilde{C}$ are the argmin of $g_{iT}'(\rho_1, C)g_{iT}(\rho_1, C)$ for i = 1, 2. Then $J_{iT} = g_{iT}'(\hat{\rho}_1, \hat{C})W_{iT}^{-1}g_{iT}(\hat{\rho}_1, \hat{C})$ for i = 1, 2 is asymptotically distributed as a chi–square random variable with six degrees of freedom.

Significance levels of the test statistics are reported in Table III. Two important results emerge here. First, we can never reject the null hypothesis that $EZ_{2t}H_t(\rho_1^o, C^o) = 0$ at the 1 per cent significance level. Second, in only one case can we reject the null hypothesis that $EZ_{1t}H_t(\rho_1^o, C^o) = 0$ at the 1 per cent significance level. This exception occurs, however, when the data set $(c, \tilde{y})$ is used over the sample period 1951.1–1985.3. Here the significance level of the test statistic of the chi–square statistic is 0.008.

Viewed as a whole, our results reveal some evidence against the CTMH. But this evidence is surprisingly weak and is consistent with the view that the empirical shortcomings of the DTMH are primarily due to the effects of temporal aggregation bias.

3. The Permanent Income Hypothesis: A General Equilibrium Model

3.A. The Model

This section describes a continuous time general equilibrium model that nests, as a special case, the CTMH. Throughout, we suppose that the time series on economy–wide consumption, the stock of capital, work effort, and output correspond to the solution of an optimal resource allocation problem which can be decentralized as a competitive equilibrium.

A representative consumer ranks alternative streams of consumption and leisure according to this preference specification:

$$(3.1) \qquad E_0 \int_0^{\infty} \exp(-rt)\{-0.5[c(t) - b(t)]^2 - \alpha(t)h(t)\}\, dt,$$

where $r > 0$ denotes the subjective rate of time preference; $b(t)$, the consumer's bliss point for consumption at time t; $c(t)$, consumption at time t; $h(t)$, hours of work, or labor, effort at time t; $\alpha(t)$, the marginal disutility of work at time t; and E_t, the expectations operator conditioned on the time t information set $I(t)$. Throughout, we assume that $b(t)$ and $\alpha(t)$ are deterministic, possibly trivial, functions of time.

An aggregate technology converts time t capital, $k(t)$, and labor effort, $h(t)$, into consumption goods:

$$(3.2) \qquad \tilde{y}(t) = \min\{\tilde{\delta}k(t),\ \tau(t)h(t)\} + e(t),$$

where $\tilde{y}(t)$ denotes time t gross output and $\tilde{\delta} > 0$. We think of the variable $e(t)$ as the time t endowment of the consumption good. It can also be interpreted as an aggregate shock to the production function which affects only the average productivity of labor and capital. The variable $\tilde{\delta}/\tau(t)$ represents the (possibly) stochastic labor requirement per unit of capital. We can interpret this labor requirement in at least two ways. One is that labor is required to maintain the capital stock, while the other is that labor is required to run it.

The economy–wide resource constraint is given by

$$(3.3) \qquad \tilde{y}(t) = c(t) + Dk(t) + \pi k(t),$$

where $\pi > 0$ is the rate at which capital depreciates and D is the time derivative operator.

Following Hansen (1987) and Sargent (1987), we do not impose a nonnegativity constraint on the choice variables of the model. Imposing that constraint makes solving the model analytically difficult, if not impossible. Instead, we follow Hansen (1987) in imposing the requirement that[6]

$$(3.4) \qquad E_0 \int_0^\infty \exp(-rt)k(t)^2\,dt < \infty.$$

This condition emerges from viewing our infinite horizon economy as the limit of a sequence of finite horizon economies in which we impose the constraint that the terminal capital stock is zero. [See Hansen and Sargent (1989).] Finally, in deriving the optimal resource allocation problem, for convenience, we impose the restriction that capital and labor are always fully utilized:

$$(3.5) \qquad \tilde{\delta}k(t) = \tau(t)h(t).$$

In Appendix A, we discuss conditions under which this restriction is nonbinding. Let $\delta = \tilde{\delta} - \pi > 0$, and define

$$(3.6) \qquad H(t) \equiv \alpha(t)\tilde{\delta}/\tau(t).$$

Here $H(t)$ is the utility cost of the labor required to make a unit of capital productive.

In displaying the solution to the planner's problem, we find the following notation convenient:

$$(3.7) \qquad x_p(t) = \delta \int_0^\infty \exp(-\delta\tau)E_t x(t+\tau)\,d\tau.$$

Definition (3.7) applies to any random variable for which the indicated conditional expectation exists. In Appendix B, we show that, for the class of e(t), b(t), and H(t) processes we consider, the equilibrium laws of motion for Dk(t) and c(t) are given by

$$
\begin{aligned}
Dk(t) &= -(\delta-r)k(t) + [e(t) - e_p(t)] - (\delta-r)\delta^{-1}e_p(t) - [b(t) - b_p(t)] \\
&\quad + (\delta-r)\delta^{-1}b_p(t) - H_p(t)/\delta, \\
c(t) &= (2\delta-r)k(t) + (2\delta-r)\delta^{-1}e_p(t) + [b(t) - b_p(t)] - (\delta-r)\delta^{-1}b_p(t) + H_p(t)/\delta
\end{aligned}
\tag{3.8}
$$

when $r < 2\delta$. When $r \geq 2\delta$, there does not exist an equilibrium for which (3.4) is satisfied.

Relations (3.8) imply that investment is an increasing function of the difference between the current endowment, e(t), and the weighted sum of current and expected future values of the endowment, $e_p(t)$. Notice also that, other things equal, investment decreases when the current utility of consumption is high, that is, $b(t) > b_p(t)$. In addition, investment depends negatively on $H_p(t)$, reflecting the utility cost of the labor input needed to make additions to the capital stock productive in the future. The impact of the remaining terms in (3.8) on Dk(t) depends on the relative magnitude of the net marginal productivity of capital, δ, and the planner's discount factor, r. These terms equal zero if $\delta = r$.

Relations (3.8) also imply that c(t) depends positively on the stock of capital, on $e_p(t)$, and on the value of b(t) relative to $b_p(t)$. In addition, c(t) depends positively on $H_p(t)$. This is because a high value of $H_p(t)$ signifies a low opportunity cost of consuming goods at time t as opposed to combining them with labor in order to produce future consumption goods. The impact of the remaining terms in (3.8) on c(t) depends on the relative magnitudes of δ and r. These terms equal zero when $\delta = r$.

We now determine conditions under which our model implies the CTMH. Let $\mu_{x_p}(t)$ denote the change in the value of $x_p(t)$ due to a disturbance in x(t) that is unpredictable on the basis of $x(t-\tau)$ for all $\tau > 0$. Equations (3.8) then imply that

$$(3.9) \qquad [D-(r-\delta)]c(t) = (2-r/\delta)[\mu_{e_p}(t) - \mu_{b_p}(t)] + \mu_{H_p}(t)/\delta$$
$$+ Db(t) + (\delta-r)b(t) - H(t).$$

[See Appendix B for a more careful characterization of $\mu_{x_p}(t)$ and a derivation of (3.9).] Since b(t) is by assumption deterministic, the level of continuous time consumption will satisfy the CTMH if and only if H(t) is deterministic and $\delta = r$. Since $\alpha(t)$ is by assumption deterministic, we conclude that c(t) satisfies the CTMH only if the time t labor requirement per unit of capital, $\bar{\delta}/\tau(t)$, is deterministic. Under these conditions, relations (3.8) imply the consumption rule, $c(t) = \delta k(t) + e_p(t) + b(t) - b_p(t) + H_p(t)/\delta$. When $H(t) \equiv 0$ for all t and $\phi = 1$, so that $b(t) - b_p(t) \equiv 0$, this reduces to the standard consumption rule discussed in the PIH literature [Hayashi (1982), eqn. (1); Sargent (1987), Chap. 12, eqn. (12)].

It is useful to compare our derivation of the MH with Hall's (1978). Both analyses make assumptions which have the implication that the real rate of interest, denominated in terms of some observable commodity, is deterministic. In particular, Hall assumes that the one–period–ahead risk–free interest rate, R_t, denominated in units of the consumption good, is constant. In our model, this interest rate is stochastic, even in those circumstances in which the CTMH is satisfied. At the same time, our model implies that the risk–free interest rate, denominated in units of labor, is deterministic, regardless of whether the CTMH holds.

To show that in our model R_t is stochastic, we combine the intertemporal Euler equations for one–period capital investment and for risk–free consumption loans. The latter is simply given by $b(t) - c(t) = \exp(-r)R_tE_t[b(t+1) - c(t+1)]$. The capital investment Euler equation can be deduced by analyzing the following variation from the optimal consumption plan. At time t, the representative consumer reduces consumption and invests the proceeds in capital for one period. At time t + 1, the net product from this investment is consumed. The marginal cost, in utility terms, of this variation is $b(t) - c(t)$

$+ E_t \int_0^1 \exp[(\delta - r)\tau] H(t+\tau)\, d\tau$. The first term is the marginal utility of consumption at time t, while the second is the utility cost of the additional amount of labor required to make the increment in capital productive. The marginal benefit of this variation is $\exp(\delta - r) E_t[b(t+1) - c(t+1)]$. This is the discounted, expected marginal utility of the increase in consumption at t + 1 made possible by the expansion of the capital stock. Along an interior optimum, these costs and benefits must be equal. Combining the resulting expression with the Euler equation for risk–free consumption loans, we obtain

$$(3.10) \qquad R_t = \frac{\exp(\delta)}{1 + E_t\{\int_0^\infty \exp[(\delta - r)\tau] H(t+\tau)\, d\tau\}/[b(t) - c(t)]}.$$

In our model, c(t) is a random variable as long as e(t) is. It follows that R_t will be stochastic even if H(t) is deterministic, as long as e(t) is stochastic.

Thus, a constant risk–free real interest rate, denominated in units of the consumption good, is not a necessary condition for the MH to hold. In contrast, the risk–free interest rate, denominated in units of labor, is given by $\exp(r)\alpha(t)/\alpha(t+1)$. Given our assumptions on $\alpha(t)$, this interest rate is deterministic.

The remainder of Section 3.A derives restrictions on the parameters of the model which guarantee that it has sensible asymptotic behavior. By sensible we mean that (i) the time zero conditional means of c(t) and k(t) converge to paths which are strictly positive and (ii) the time zero conditional mean of c(t) converges to a path which is strictly less than the time zero conditional mean of the bliss point b(t). Given the linearity of our model, we can examine this problem abstracting from uncertainty.

The deterministic analogs of the laws of motion of e(t), b(t), and H(t) described in Sections 3.B and 3.C below are

$$(3.11) \qquad e(t) = e \times \exp(\theta t), \quad b(t) = b \times \exp(\theta t), \quad H(t) = H \times \exp(\theta t),$$

where e, b > 0; H, e < b; $\theta \geq 0$; and $r - 2\theta > 0$. The last restriction on r and θ, together with (3.8), implies that $\delta > \theta$. Relations (3.8) and (3.11) imply that the equilibrium laws of motion for k(t) and c(t) are given by

$$k(t) = \begin{cases} k(0)\exp[(r-\delta)t] + \dfrac{(b-e)(\theta-r+\delta) - H}{(\delta-\theta)(\theta-r+\delta)}\{\exp(\theta t) - \exp[(r-\delta)t]\} & (\theta \neq r-\delta) \\ [k(0) - Ht/(\delta-\theta)]\exp(\theta t) & (\theta = r-\delta) \end{cases}$$

(3.12)

$$c(t) = \begin{cases} b \times \exp(\theta t) + (2\delta-r)k(0)\exp[(r-\delta)t] - \dfrac{H}{(\theta-r+\delta)}\exp(\theta t) & \\ \qquad + (r-2\delta)\dfrac{(b-e)(\theta-r+\delta) - H}{(\delta-\theta)(\theta-r+\delta)}\exp[(r-\delta)t] & (\theta \neq r-\delta) \\ \{(2\delta-r)[k(0) + (e-Ht)/(\delta-\theta)] + H/(\delta-\theta)\}\exp(\theta t) & (\theta = r-\delta). \end{cases}$$

Here k(0) is the initial capital stock.

From (3.12), we can see that our criteria for sensible asymptotic behavior are satisfied only when $r - \delta \leq \theta$. If this condition does not hold, then two things are possible, depending on the relative magnitude of k(0) and $\kappa \equiv b - e + H/(r-\delta-\theta) > 0$. If $k(0) \geq \kappa$, then c(t) eventually exceeds the bliss point; if $k(0) < \kappa$, then c(t) eventually becomes negative. Suppose that $r - \delta = \theta$. Then k(t) and c(t) converge to a strictly positive growth path if and only if H = 0. In addition, c(t) converges to a path strictly below the bliss point as long as $b > (2\delta-r)[k(0) + e/(\delta-\theta)]$. Finally, suppose that $r - \delta < \theta$. Here c(t) converges to a path which is strictly positive if and only if $H/(\theta-r+\delta) < b$ and is strictly less than the bliss point if and only if $0 < H/(\theta-r+\delta)$. In addition, to guarantee that k(t) converges to a strictly positive growth path, we require that $H/(\theta-r+\delta) < b - e$.

To summarize, the only two cases that are consistent with our criteria for sensible asymptotic behavior are

(3.13) $\quad r - \delta < \theta$ and $0 < \dfrac{H}{\theta - r + \delta} < b - e$

and

(3.14) $\quad r - \delta = \theta$, $H = 0$, and $b > (2\delta - r)[k(0) + e/(\delta - \theta)]$.

Notice that, when $H = 0$, equations (3.14) imply that the model has sensible asymptotic properties only in the knife–edge case, $r - \delta = \theta$. If $\theta = 0$, this requires that $r = \delta$. Indeed, this restriction, or its discrete time analog, is typically imposed in capital accumulation models of this sort [for example, Hansen (1987) and Sargent (1987)]. This restriction is also the analog to the one imposed by Flavin (1981) and Hayashi (1982), who require that the constant real interest rate equal the representative agent's subjective rate of time preference.

Relations (3.13) imply that, when $H > 0$, the model does not have this knife–edge property. For example, in this case, δ can exceed r. To see why, suppose that $\theta = 0$. Then the nonstochastic steady–state level of consumption is $b - H/(\delta - r)$. Evidently, if $H = 0$, consumption simply converges to the bliss point. When $H > 0$, however, the labor required to make the capital stock productive has disutility associated with it. This disutility reduces the optimal steady–state capital below the level needed to sustain steady–state consumption at the bliss point. Similarly, with H and θ greater than zero, the model can accommodate values of δ less than r.

3.B. The Deterministic Labor Requirement Model

Now we describe a parameterization of the model in which $H(t)$ is nonstochastic. As a special case, when $r = \delta$, continuous time consumption satisfies the MH. We call this the deterministic labor requirement (DLR) model.

Our parameterization of the underlying forcing variables in the economy implies that consumption and output grow at the same geometric rate over time. While this parameterization is very restrictive, it does have an important compensating advantage: it implies that our model applies to consumption and output data which have been detrended with a procedure that assumes a common geometric trend. This allows us to accommodate growth in an internally consistent way while preserving the applicability of a set of econometric tools developed for nongrowing time series.

We suppose that b(t) is nonstochastic and satisfies (3.11). Since the level of continuous time consumption satisfies the MH if and only if H(t) is nonstochastic, we suppose that H(t) is nonstochastic and also satisfies (3.11).

The shock to endowment income is assumed to satisfy

(3.15) $$e(t) = e_1(t) + e_2(t),$$

where

(3.16) $$De_i(t) = e_i \times \exp(\theta t) + \frac{\eta_i(t)}{a_i + D},$$

where $a_i > 0$ for $i = 1, 2$ and $a_1 \neq a_2$. Let $x(t) = [\eta_1(t)\eta_2(t)]'$. The vector x(t) is the continuous time linear least squares innovation to the joint $[e_1(t)e_2(t)]$ process and satisfies

(3.17) $$E[x(t)x(t-u)]' = \exp(2\theta t)\xi(u)\overline{V}$$

for all real values of u. Here $\xi(u)$ is the Dirac delta generalized function and $\overline{V}$ is a 2×2 positive definite symmetric matrix of constants. Thus, e(t) is the sum of two stochastic processes with first derivatives that are AR(1) continuous time stochastic processes. The reason for assuming that the endowment process is the sum of two stochastic processes, the realizations of which are separately observed by agents, is to guarantee that the observed bivariate consumption and income process is of full spectral rank.

Define y(t) as net output; that is, $y(t) = \bar{y}(t) - \pi k(t)$. According to our specification, all deterministic terms and innovation standard deviations grow at the same rate ϕ. Thus, not surprisingly, we can detrend c(t) and $c(t) - y(t)$ by ϕ^t to obtain a stationary stochastic process. Define

(3.18) $\overset{*}{q}(t) = [\overset{*}{c}(t) - \overset{*}{y}(t), (D+\theta)\overset{*}{c}(t)]'$,

where $\overset{*}{c}(t) = c(t)\exp(-\theta t)$ and $\overset{*}{y}(t) = y(t)\exp(-\theta t)$. In deriving the reduced-form representation for $\overset{*}{q}(t)$, note that relations (3.8) and our definition of $y(t)$ imply that

$$[D-(r-\delta)][c(t)-y(t)] = \frac{2\delta - r}{\delta} De_p(t) - De(t) - \frac{2\delta - r}{\delta} Db_p(t) + Db(t) + DH_p(t). \tag{3.19}$$

It follows from (3.9), (3.16), and (3.19) that $\overset{*}{q}(t)$ has the continuous time, scalar, autoregressive vector moving average (SARMA) representation

$$[D-(r-\theta-\delta)]\overset{*}{q}(t) = B_c(D+\theta)X(t) + T_c, \tag{3.20}$$

where

$$B_c(D) = [b_{ij}(D)] \qquad (i, j = 1, 2)$$

$$b_{1j}(D) = \{(\delta-r)(a_j+\delta) + (2\delta-r)D\}/\{(2\delta-r)(a_j+D)\} \qquad (j = 1, 2)$$

$$b_{2j}(D) = D \qquad (j = 1, 2) \tag{3.21}$$

$$X(t) = [(2\delta-r)\delta^{-1}\eta_1(t)/(a_1+\delta), (2\delta-r)\delta^{-1}\eta_2(t)/(a_2+\delta)]'$$

$$E[X(t)X(t-u)]' = \xi(u)V_c.$$

In (3.20), T_c is a two-dimensional vector of constants and V_c is a 2×2 positive definite symmetric matrix of constants.

Notice that, when $\delta = r$, nondetrended consumption satisfies the MH. Detrended consumption does not, though, since relations (3.20) and (3.21) imply that, when $\delta = r$, $\overset{*}{c}(t)$ will have a nonzero continuous time autoregressive root equal to θ.

3.C. The Stochastic Labor Requirement Model

Now we describe a parameterization of the model in which the labor requirement per unit of capital, $\tau(t)$, is stochastic, so that the CTMH does not hold even when $\delta = r$. We call this the stochastic labor requirement (SLR) model.

Our specification of $b(t)$ and $\alpha(t)$ is the same as above. Now, however, we assume that $H(t)$ is a continuous time AR(1) random variable with time–varying drift:

$$(3.22) \qquad H(t) = \bar{H} \times \exp(\theta t) + \epsilon(t)/(f+D),$$

where $f > 0$, $E[\epsilon(t)\epsilon(t-u)] = \exp(2\theta)\xi(u)\sigma_\epsilon^2$, and $\sigma_\epsilon^2 > 0$. Since $H_t = \bar{\delta}\alpha(t)/\tau(t)$, relation (3.22) implies that the labor requirement per unit of capital is stochastic.

The shock to endowment income, $e(t)$, is assumed to satisfy

$$(3.23) \qquad De(t) = \bar{e} \times \exp(\theta t) + \eta(t)/(a+D),$$

where $E[\eta(t)\eta(t-u)] = \exp(2\theta t)\xi(u)\sigma_\eta^2$ and $\sigma_\eta^2 > 0$. Let $\mathbf{x}(t) = [\epsilon(t)\eta(t)]'$. The vector $\mathbf{x}(t)$ is the continuous time linear least squares innovation to the joint $[H(t)De(t)]$ process and satisfies

$$(3.24) \qquad E[\mathbf{x}(t)\mathbf{x}(t-u)]' = \exp(2\theta t)\xi(u)\overline{\mathbf{V}},$$

where $\overline{\mathbf{V}}$ is a 2×2 positive definite symmetric matrix of constants.

Relations (3.9) and (3.22)–(3.24) imply that $[D - (r-\theta-\delta)]\overset{*}{q}(t)$ has the continuous time SARMA representation

$$(3.25) \qquad [D - (r-\theta-\delta)]\overset{*}{q}(t) = \mathbf{B_c}(D+\theta)\mathbf{X}(t) + \mathbf{T_c},$$

where

$$\mathbf{B_c}(D) = [b_{ij}(D)] \qquad (i, j = 1, 2)$$

$$
(3.26)\quad
\begin{aligned}
&b_{11}(D) = [(\delta-r)(a+\delta) + (2\delta-r)D]/(a+D)\\
&b_{12}(D) = D/(f+D)\\
&b_{21}(D) = (2\delta-r)D\\
&b_{22}(D) = D(D-\delta)/(f+D)\\
&X(t) = [\delta^{-1}\eta(t)/(a+\delta),\ \epsilon(t)/(f+\delta)]'\\
&E[X(t)X(t-u)]' = \xi(u)V_c.
\end{aligned}
$$

In (3.25), T_c is a two-dimensional vector of positive constants, and in (3.26), V_c is a 2×2 dimensional positive definite symmetric matrix of constants. Relations (3.25) and (3.26) imply that, unlike the DLR model described in Section 3.B, here c(t) does not satisfy the CTMH even when $\delta = r$.

4. Empirical Tests of the Structural Continuous Time Models

In this section, we estimate and test our two continuous time structural models. First we discuss our empirical methodology, and then we report our empirical results.

4.A. The Estimation Strategy

In Section 3, we derived the constrained continuous time SARMA representations for the vector $\overset{*}{q}(t)$ implied by our two continuous time models. [See (3.20)–(3.21) and (3.25)–(3.26).] To proceed with estimation, we must deduce the implications of these SARMA representations for the probability law of the vector of observable variables.

While $\overset{*}{q}(t)$ is defined in terms of detrended consumption and net output, we actually estimated our model using data on gross output. This decision was based on two considerations. One is that the data on aggregate depreciation is not particularly reliable. The other is that our model of depreciation is not likely to be consistent with the model of depreciation used by the U.S. Department of Commerce.

Because of our decision to use GNP rather than net national product (NNP) data in our empirical work, we must derive the implications for the analog to $\overset{*}{q}(t)$ defined in terms of detrended consumption and gross output. Define $\bar{q}(t)$ to be the 2×1 continuous time stochastic process which has as its first element the difference between detrended quarterly averaged consumption and gross output and its second element the detrended first difference of averaged consumption. The vectors $\overset{*}{q}(t)$ and $\bar{q}(t)$ differ in two important respects. First, $\overset{*}{q}(t)$ involves a measure of detrended NNP, whereas $\bar{q}(t)$ involves a measure of detrended GNP. Second, $\overset{*}{q}(t)$ represents point–in–time measured variables, whereas $\bar{q}(t)$ represents variables averaged over the discrete data sampling interval.

In Appendix C, we derive the following linear mapping between $\overset{*}{q}(t)$ and $\bar{q}(t)$:

(4.1) $$\overset{*}{q}(t) = H(D+\theta)G(D+\theta)^{-1}\,\bar{q}(t),$$

where

(4.2) $$G(D) = [1-e^{-D}]/D\begin{bmatrix} 1 & 0 \\ 0 & (1-e^{-D})/D \end{bmatrix} \quad \text{and} \quad H(D) = \begin{bmatrix} D & 0 \\ 0 & D+\pi \end{bmatrix}/(D+\pi).$$

Substituting (4.1) and (4.2) into (3.20) and (3.25), we obtain the time series representations for $\bar{q}(t)$ implied by the two versions of our model.

The assumptions we have imposed on the structural parameters of the model are sufficient to guarantee that $\bar{q}(t)$ is a covariance stationary stochastic process with conditionally homoscedastic disturbances. Define

(4.3) $$Q(t) = \bar{q}(t) - E\bar{q}(t).$$

Suppose we have a sample on $Q(t)$ for $t = 1, 2, \ldots, T$. Our estimation criterion is the frequency domain approximation to the Gaussian density function suggested by Durbin (1961), Hannan (1970), and Hansen and Sargent (1980). This criterion requires that we compute the theoretical spectral density matrix of the discrete process $\{Q(t),\ t \text{ integer}\}$ at frequency ω, which we denote $Z(\omega)$. We accomplish this in two steps. First, using results in Phillips (1958), it can be shown that the spectral density matrix of $\{Q(t),\ t \text{ real}\}$ implied by the model of Section 3.B and equations (3.20) and (4.1) is given by

$$(4.4) \qquad Z^c(\omega) = \psi(i\omega+\theta)A_c(i\omega+\theta)^{-1}\, V_c[A_c(i\omega+\theta)']^{-1}\, \psi(i\omega+\theta)'$$

for $-\infty \leq \omega \leq \infty$, where $\psi(s) = G(s)H(s)^{-1}$ and $A_c(s) = [s-(r-\delta)]B_c(s)^{-1}$. The corresponding spectral density implied by the model of Section 3.C is

$$(4.5) \qquad \mathbf{Z}^{\mathbf{c}}(\omega) = \psi(i\omega+\theta)\mathbf{A}_{\mathbf{c}}(i\omega+\theta)^{-1}\, \mathbf{V}_{\mathbf{c}}[\mathbf{A}_{\mathbf{c}}(i\omega+\theta)']^{-1}\, \psi(i\omega+\theta)',$$

where $\mathbf{A}_{\mathbf{c}}(s) = [s-(r-\delta)]\mathbf{B}_{\mathbf{c}}(s)^{-1}$. Second, Hannan (1970, p. 45) shows that the following "folding operator" links $Z(\omega)$ and $Z^c(\omega)$:

$$(4.6) \qquad Z(\omega) = \sum_{k=-\infty}^{\infty} Z^c(\omega + 2\pi k).$$

Equation (4.6) together with (4.4) or (4.5) provides a computationally feasible algorithm for obtaining $Z(\omega)$, for a given ω, from $[\psi, A_c, V_c]$ or $[\psi, \mathbf{A}_{\mathbf{c}}, \mathbf{V}_{\mathbf{c}}]$. Because this algorithm is relatively slow, we used an alternative method based on a partial fractions decomposition of Z^c. [See Durbin (1961); Hannan (1970), pp. 405–407; and Hansen and Sargent (1980).]

The preceding strategy assumes that the values of $\phi = \exp(\theta)$ and $E\tilde{q}(t)$ are known.[7] By assumption, $\theta > r-\delta$. Thus, c(t) and y(t) have asymptotic growth rates equal to ϕ. In practice, we fixed the value of ϕ at exp(0.004568). (See Section 2.A for a description of the growth properties of our measures of consumption and output.) Given ϕ, we formed time series on $\tilde{q}(t)$ and Q(t). The first series was calculated using the measure of consumption denoted by c and $\tilde{y}$ described in Section 2. The second series was computed using the demeaned values of $\tilde{q}(t)$. Throughout, we fixed the value of the parameter r at 0.0098, which implies an annual rate of time preference of 4 per cent. Consequently, the free parameters of the model of Section 3.B are a_1, a_2, δ, d, and the three independent elements of V_c, while the free parameters of the model of Section 3.C are a, f, δ, d, and the three independent elements of V_c.

An implication of the results in Christiano and Marshall (1987) is that both of our models give rise to constrained SARMA(4,5) representations for {Q(t), t integer}. This SARMA is characterized by a fourth–order scalar polynomial E(L), a 2 × 2–order matrix polynomial C(L), and a 2 × 2 positive semidefinite matrix V^d which must satisfy

$$(4.7) \qquad Z(\omega) = C(e^{-i\omega})V^d C(e^{i\omega})'/[E(e^{-i\omega})E(e^{i\omega})].$$

We imposed the normalization C(0) = I, det[C(z)] = 0 implies $|z| > 1$, and E(0) = 1. The algorithm we used to calculate E, C, and V^d is the one described by Rozanov (1967), Chap. I, Sec. 10.

4.B. The Results

In this section, we report the results of estimating four versions of our model. These specifications are the DLR and SLR models, both with and without imposing the constraint that $\delta = r$. Our estimates for the two versions of the DLR model and the SLR model are displayed in Table IV.

Both the DLR and SLR models imply reasonable asymptotic behavior for the stock of capital and consumption only when conditions (3.13) or (3.14) hold. Without these constraints, the constants in our model are not identified. However, restrictions (3.13) and (3.14), in conjunction with our point estimates and the assumed value of r, imply that H > 0. To see this, suppose first that H = 0. Then equations (3.14) require that $\delta = r - \theta = 0.005232$. Our point estimates of δ are 0.0088 and 0.0078 in the DLR and SLR models, respectively, with corresponding standard errors of 0.0015 and 0.0007. This means that if we insist on assuming that H = 0 and r = 0.0098, then the estimated models have the perverse implication that consumption fluctuates about the bliss point. Now suppose that $H > 0$. Restrictions (3.13) require that $\delta > 0.005232$, which is easily satisfied by both models.

We now assess the overall empirical performance of these models. The most parsimoniously parameterized unconstrained model which nests our four structural models is a SARMA(3,4).[8] Let J_T denote twice the difference between the maximized log likelihoods for the unconstrained and the constrained SARMA specifications. Then J_T is asymptotically distributed as a chi–square random variable with degrees of freedom equal to the number of restrictions imposed in the constrained specification. Without constraints on δ, there are 16 degrees of freedom; with δ constrained to equal r, there are 15. The J_T statistic can be multiplied by an adjustment factor suggested by Whittle (1953), Lissitz (1972), and Sims (1980) that is designed to correct for small sample bias. We denote this adjusted test statistic by J_T^*.[9]

According to the values of J_T and J_T^* displayed in Table IV, all four models perform fairly well. First, the adjusted likelihood ratio statistics fail to reject any of the four models at the 5 per cent marginal significance level. In fact, neither of the SLR models can be rejected at even the 10 per cent significance level. Second, according to the unadjusted likelihood ratio statistics, neither of the SLR models or the constrained DLR model can be rejected at the 5 per cent significance level.

The large number of parameters in the unconstrained SARMA(3,4) representation raises questions about the power of the specification tests. We can construct a more powerful test of the constrained DLR and SLR models because both are nested within an unconstrained SARMA(2,3) specification. (Against this alternative, there are 11 degrees of freedom.) Not surprisingly, when we do this, we find more evidence against both models. As reported in Table IV, both the unadjusted and adjusted likelihood ratio statistics imply that both versions of the DLR and SLR models can be rejected at the 5 per cent significance level, but not at the 1 per cent level. Although the unconstrained models are not nested in the SARMA(2,3) specification, for completeness we did similar calculations for these models. (With δ unconstrained, there are 10 degrees of freedom.) Table IV includes the results of these calculations.

Overall, our goodness-of-fit tests do not yield overwhelming evidence against any of the structural models. As an additional diagnostic, we examine the implications of the different models for a vector autoregressive representation (VAR) of the data. Column (1) of Table V reports an unconstrained VAR(2), estimated by least squares. We report a second-order VAR because a likelihood ratio test failed to reject, at even a 10 per cent significance level, that the third lag is zero. The VARs implied by our structural models are reported in columns (2)–(5) of Table V. In principle, all of these VARs are infinite ordered. We use the truncation rule of not reporting matrix coefficients which have a maximal element smaller than 0.03 in absolute value.

Consider first the VAR generated by the constrained DLR model [column (2)]. While this model implies that the level of consumption satisfies the CTMH, it does not imply that actual measured consumption is uncorrelated with lagged values of consumption or output. In Section 2, we found very strong evidence against the DTMH, but relatively little evidence against the CTMH. The results in Table V suggest that the effect of temporal aggregation discussed by Working (1960) is the major factor accounting for the reasonably good performance of the constrained DLR model. To see this, notice first that the (2,2) element of the coefficient matrix on the first lag of the constrained VAR (which is the coefficient on the own first lag of $\phi^{-t}\Delta c_t$) is approximately 0.27. This is within one standard error of the point estimate (0.313) of the corresponding coefficient in the unconstrained VAR. Moreover, this point estimate is more than three standard errors away from zero. In a model which implied the DTMH, this coefficient would equal zero.

To complete our argument, notice that the nonzero values of the coefficients on lagged values of $\phi^{-t}(c_t - \tilde{y}_t)$ in the second row of the VAR implied by the constrained DLR model also reflect the effects of time averaging. However, this effect is harmful with respect to the overall fit of the constrained DLR model. This is because the sign on $\phi^{-t}(c_{t-1} - y_{t-1})$ in the equation for $\phi^{-t}\Delta c_t$ is positive in the constrained VAR, while it is negative in the unconstrained VAR. Since the latter coefficient is not precisely estimated, this effect is not sufficiently important to negate the favorable impact of the effects suggested by Working (1960).

Finally, note that the VARs in columns (3)–(5) are very similar to the one in column (2). This is consistent with the evidence in Table IV according to which the likelihood values of all the structural models are quite similar.

5. Concluding Remarks

In this paper, we developed and tested simple variants of the PIH model which are consistent with the fact that measured aggregate U.S. consumption does not behave like a martingale. We investigated a variety of reasons why lagged consumption and output help predict the change in measured aggregate quarterly U.S. consumption. Two reasons received particular attention. One is that the MH holds in the (unobserved) continuous time consumption process, with serial persistence in measured consumption being an artifact of temporal aggregation. Using atheoretical econometric methods, we found much less evidence against the CTMH than against the DTMH. We followed up on this by estimating and testing a particular continuous time general equilibrium model which nests the CTMH as a special case. The evidence against this model is far from overwhelming. This suggests that the MH may yet be a useful way to conceptualize the relation between aggregate quarterly U.S. consumption and output. The other reason we focused on is that exogenous shocks to the economic system generate serial persistence in the first difference of consumption. Using a simple continuous time general equilibrium setup, we modeled this shock as a stochastic perturbation to the amount of labor required to make capital productive. Again, we found little evidence against this model.

On the basis of aggregate quarterly U.S. consumption and output data, convincingly distinguishing between the different continuous time models we considered is difficult. However, the continuous time martingale model does have implications which we did not test, but which call into question its plausibility. One such implication is that the capital/labor ratio is deterministic. This implication is clearly counterfactual. While this problem could be remedied by allowing for measurement error, we regard our stochastic labor requirement model as a more promising starting point for future research.

Throughout, we restricted our empirical work to aggregate consumption data that are measured quarterly. This is the frequency of data used in most studies of the PIH [for example, Hall (1978) and Flavin (1981)]. In addition, we implemented our structural models using data on GNP. While a variety of measures of consumption are available monthly, GNP data are not. In principle, this difficulty could be circumvented by deriving the likelihood function for monthly consumption and quarterly GNP data. However, an important maintained assumption of our structural models is that measured consumption and output have the same growth rate. Our measure of consumption is not available monthly, and existing monthly measures of consumption appear to have a very different growth rate than GNP does. (See footnote 4.)

At the same time, we recognize that the monthly consumption data are problematic for the CTMH. This is because the first difference of monthly consumption, as measured by real expenditures on nondurables plus services, is negatively autocorrelated. [See, for example, Heaton (1989).] If the CTMH holds, and monthly and quarterly consumption were measured with the same degree of accuracy, then the first difference of both monthly and quarterly consumption should display the same autocorrelation coefficients. Heaton (1989) considers a model in which agents who have time–nonseparable preferences over alternative streams of consumption make decisions on a continuous time basis. His model has the attractive feature that it can simultaneously match the estimated autocorrelation function of the first difference of both monthly and quarterly consumption, as measured by expenditures on nondurables and services. Unfortunately, this particular consumption measure is problematic from the perspective of our equilibrium model because this measure and real GNP appear to have very different growth rates. Nevertheless, we are sympathetic to the basic strategy of adopting specifications of preferences and technology which can in principle explain the time series properties of data measured at different sampling intervals.

APPENDIX A

Ruling Out Underutilization of Capital and Labor

Our decision rules in Section 3 solve an optimum problem subject to the constraint that capital and labor are never underutilized [equation (3.5)]. Here we discuss conditions under which this constraint is nonbinding.

That labor is never underutilized follows trivially from the facts that $\tau(t)$ is known when $Dk(t)$ and $h(t)$ are chosen and that $\alpha(t) > 0$. Thus, $\bar{\delta}k(t) \geq \tau(t)h(t)$ for almost all t.

It is harder to establish general conditions under which the constraint that capital is never underutilized is nonbinding. For this reason, we restrict ourselves to finding conditions that are sufficient in deterministic steady state. In addition, our conditions only guarantee that local deviations from the constrained optimum which violate (3.5) are suboptimal. We consider two alternative deviations: (i) the capital stock is increased without adjusting hours, or (ii) hours worked are decreased without changing the path of the capital stock.

Consider (i). Here we use an argument analogous to our derivation of the real rate of interest in Section 3. Suppose that investment is increased at time t by an equal reduction in $c(t)$ and that the increased stock of capital is held until $t + \Delta$, at which time the undepreciated part is consumed. The marginal cost of this is $b(t) - c(t)$, while the discounted marginal benefit is $\exp[-(r+\pi)\Delta][b(t+\Delta) - c(t+\Delta)]$. The condition that (i) is locally suboptimal requires that

$$b(t) - c(t) > \exp[-(r+\pi)\Delta][b(t+\Delta) - c(t+\Delta)] \tag{A.1}$$

for all $t \geq 0$ and $\Delta > 0$. According to (3.12) and the fact that $r - \delta \leq \theta$ [by equations (3.13) and (3.14)], we have that, in steady state, $b(t+\Delta) - c(t+\Delta) = \exp(\theta\Delta)[b(t) - c(t)]$. Therefore, in steady state, (A.1) is equivalent to

(A.2) $\quad r + \pi > \theta.$

Now consider (ii). Decreasing hours worked without adjusting capital generates instantaneous costs and benefits equal to $\tilde{\delta}\tau(t)[b(t) - c(t)]$ and $\alpha(t)$, respectively, which produces a local decrease in utility if

(A.3) $\quad b(t) - c(t) > H(t)/\tilde{\delta}.$

In steady state, (A.2) implies (A.3). To see this, note that the deterministic version of equation (3.9) is $[D - (r-\delta)][b(t) - c(t)] = H(t)$ or, in steady state,

(A.4) $\quad b(t) - c(t) = H(t)/[\theta - (r-\delta)]$

since in this case $D[b(t) - c(t)] = \theta[b(t) - c(t)]$. Equation (A.3) follows immediately from (A.4) after we use (A.2) and the fact that $\delta = \tilde{\delta} - \pi$.

We conclude that if (A.2) is satisfied, then (3.5) is locally nonbinding in steady state. Condition (A.2) is satisfied for the models in the text since $\pi > 0$ and we parameterize r and θ in such a way that $r > \theta$.

APPENDIX B

Deriving the Decision Rules (3.8) and (3.9)

Here we show how we derive the decision rules (3.8) and (3.9).

Proceeding as do Hansen and Sargent (1980), we can show that the Euler equation for the social planner's problem is

$$\text{(B.1)} \qquad [D-\delta][D-(r-\delta)]k(t) = [D-(r-\delta)][e(t)-b(t)] + H(t).$$

The unique solution to this problem which satisfies (3.4) is

$$\text{(B.2)} \qquad [D-(r-\delta)]k(t) = e(t) - b(t) - (2\delta-r)E_t \int_0^\infty e^{-\delta\tau}\{e(t+\tau) - b(t+\tau)\}\, d\tau$$
$$- E_t \int_0^\infty e^{-\delta\tau} H(t+\tau)\, d\tau.$$

When definition (3.7) is taken into account, (B.2) produces the first part of (3.8). The second part of (3.8) is obtained by substituting the first into the relation $c(t) = \delta k(t) - Dk(t) + e(t)$.

To derive (3.9), we first present some preliminary results regarding $x_p(t)$. Suppose the fundamental representation for $x(t)$ is $x(t) = C(D)\epsilon(t)$. Here $C(s) = 0$ implies that Real(s) ≤ 0 and the poles of $C(s)$ lie in the closure of the left side of the complex plane. Then

$$\begin{aligned} x_p(t) &= -\delta E_t[D-\delta]^{-1}x(t) \\ \text{(B.3)} \qquad &= -\delta E_t C(D)[D-\delta]^{-1}\epsilon(t) \\ &= -\delta[C(D) - C(\delta)][D-\delta]^{-1}\epsilon(t) \end{aligned}$$

by a formula due to Hansen and Sargent (1980). Multiply (B.3) by $D - \delta$ and rearrange to obtain

$$\text{(B.4)} \qquad (D-\delta)x_p(t) + \delta x(t) = \mu_{x_p}(t),$$

where $\mu_{x_p}(t) = \delta C(\delta)\epsilon(t)$. From the second part of (3.8),

$$\text{(B.5)} \qquad [D-(r-\delta)]c(t) = (2\delta-r)[D-(r-\delta)]k(t) + [D-(r-\delta)]\{(2\delta-r)\delta^{-1}e_p(t) + [b(t)-b_p(t)] - (\delta-r)\delta^{-1}b_p(t) + H_p(t)/\delta\}.$$

Relation (3.9) follows by substituting the first part of (3.8) into (B.5) and then using (B.4).

APPENDIX C

Deriving the Probability Law for $\tilde{q}(t)$

Our strategy for obtaining the probability law for $\tilde{q}(t)$ is to derive the linear mapping relating $\overset{*}{q}(t)$ and $\tilde{q}(t)$ and then use this expression to substitute out for $\overset{*}{q}(t)$ in terms of (3.20) and (3.25).

We proceed by first obtaining the linear mapping between undetrended $\overset{*}{q}(t)$ and undetrended $\tilde{q}(t)$. Let z(t) denote the undetrended process underlying $\overset{*}{q}(t)$; that is, $z(t) = [c(t) - \tilde{y}(t),\ Dc(t)]'$. Also, let $\bar{z}(t)$ denote the undetrended, averaged data underlying $\tilde{q}(t)$; that is, $\bar{z}(t) = \exp(\theta t)\tilde{q}(t)$. Formally,

$$\text{(C.1)} \qquad \bar{z}(t) = \begin{bmatrix} \int_0^1 [c(t-\tau) - \tilde{y}(t-\tau)]\ d\tau \\ \int_0^1 [c(t-\tau) - c(t-1-\tau)]\ d\tau \end{bmatrix} = \begin{bmatrix} \int_0^1 [c(t-\tau) - \tilde{y}(t-\tau)]\ d\tau \\ \int_0^1 \left[\int_0^1 Dc(t-\tau-\mu)\ d\mu\right] d\tau \end{bmatrix}.$$

Here we have used the fact that $\int_0^1 Dc(t-\mu)\ d\mu = c(t) - c(t-1)$. Treating measured consumption and income as unit integrals of the underlying instantaneous quantities is a rough approximation to the methods used by the U.S. Department of Commerce. In operator notation:

$$\text{(C.2)} \qquad \bar{z}(t) = G(D)z(t),$$

where

$$\text{(C.3)} \qquad G(D) = [1-e^{-D}]/D \begin{bmatrix} 1 & 0 \\ 0 & (1-e^{-D})/D \end{bmatrix}.$$

In deriving (C.2), we have used the fact that $\int_0^1 x(t-\tau)\ d\tau = \int_0^1 e^{-\tau D}x(t)\ d\tau = [(1-e^{-D})/D]x(t)$.

Let $q(t)$ denote the undetrended value of $\overset{*}{q}(t)$; that is, $q(t) = \exp(\theta t)\overset{*}{q}(t) = [c(t)-y(t), Dc(t)]$. In operator notation, the link between $q(t)$ and $z(t)$ is given by

(C.4) $$q(t) = H(D)z(t),$$

where

(C.5) $$H(D) = \begin{bmatrix} D & 0 \\ 0 & D+\pi \end{bmatrix} /(D+\pi).$$

Equation (C.4) can be derived as follows: $q(t) = [c(t)-y(t), Dc(t)]' = [-Dk(t), Dc(t)]' = H(D)[-(D+\pi)k(t), Dc(t)]' = H(D)\tilde{q}(t)$, where the last equality follows from (3.3) and (C.5).

Substituting (C.4) into (C.2), we obtain

(C.6) $$q(t) = H(D)G(D)^{-1}\,\bar{z}(t)$$

which provides a mapping between the continuous time process $q(t)$ and $\bar{z}(t)$, that is, between undetrended $\overset{*}{q}(t)$ and undetrended $\tilde{q}(t)$. Finally, the link between $\overset{*}{q}(t)$ and $\tilde{q}(t)$ is obtained by multiplying both sides of (C.6) by $\exp(-\theta t)$:

(C.7) $$\overset{*}{q}(t) = H(D+\theta)G(D+\theta)^{-1}\,\tilde{q}(t),$$

which is equation (4.1) in the paper. Substituting (C.7) into (3.20) and (3.25), we obtain the time series representations for $\tilde{q}(t)$ implied by the two versions of our model.

NOTES

[1]This measure is a revised and updated version of the measure discussed in Musgrave (1979).

[2]This measure is that labeled by the data mnemonic NPT in the Wharton Econometrics data base.

[3]When this restriction is not imposed, the measured growth rates of c and $\tilde{y}$ are 0.004582 and 0.004606, respectively.

[4]In contrast to our measures c and $\tilde{y}$, the per capita growth rates of consumption and income in these latter two data sets are quite different. Over the sample period 1950.2–1985.3, the growth rates of $c_{nd} + c_s$, c_{nd}, and y_d are 0.004987, 0.003076, and 0.005502, respectively.

[5]Here W_T is a consistent estimate of the spectral density of $H_t(\rho_1^o, C^o)$ at frequency zero. In computing W_T, we took into account that $H_t(\rho_1^o, C^o)$ is autocorrelated at lag 1 but not higher. This is implied by the fact that $E_{t-2}H_t(\rho_1^o, C^o) = 0$. See Hansen, Heaton, and Ogaki (1989) for a discussion of the efficiency gains associated with imposing the exact autocorrelation structure on error terms in GMM estimation problems. This procedure was used in all weighting matrix calculations carried out in the context of tests of the CTMH in this section.

[6]Hansen's (1987) model is formulated in discrete time and sets $\alpha(t) = 0$.

[7]There is a potential incompatibility between the procedure used to estimate θ and the Gaussian maximum likelihood procedure used to estimate the remaining parameters. In estimating θ, we take logarithms of variables (in Section 2.A) which are subsequently assumed to have a Gaussian distribution.

[8]In general, the structural models with δ unconstrained imply a SARMA(4,5) representation for the observable Q(t) vector. However, the point estimates of a_2 (in the DLR model) and a (in the SLR model) are extremely large, so that the $e_2(t)$ and e(t) processes are virtually indistinguishable from continuous time random walks. Therefore, in

the SARMA(4,5) representations implied by these models, the MA matrix coefficients on the fifth lag and the AR coefficient on the fourth lag are approximately zero. By a similar argument, the models with δ constrained to equal r can be nested within a SARMA(2,3).

[9]Whittle's (1953) correction for small sample bias is as follows: Let N = the total number of parameters under the alternative hypothesis (excluding the covariance matrix of the observables), M = the number of equations, T = the number of observations, and J_T = the unadjusted likelihood ratio statistic. The adjusted statistic J_T^* is then given by $J_T^* = (1 - N/MT)J_T$. When the unconstrained alternative is a SARMA(3,4), N = 19, M = 2, and T = 141, so $J_T^* = 0.933J_T$.

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TABLE I

Results of Tests of Correlation Between the Detrended First Difference of Consumption and Lagged Values of Itself and Other Variables

Other Lagged Variables	Sample Period	Significance Levels of Zero Correlation[a]					
		Lags 1–4			Lags 2–4		
		$c, \tilde{y}$	$c_{nd} + c_s, y_d$	c_{nd}, y_d	$c, \tilde{y}$	$c_{nd} + c_s, y_d$	c_{nd}, y_d
Detrended Income	1951.3–85.3	.001	.002	.008	.029	.111	.013
	1952.3–85.3	.000	.001	.009	.029	.157	.007
	1951.3–79.1	.013	.007	.066	.159	.142	.105
Detrended Consumption Minus Income	1951.3–85.3	.002	.003	.013	.072	.141	.017
	1952.3–85.3	.000	.001	.015	.086	.220	.056
	1951.3–79.1	.018	.019	.084	.240	.149	.080

[a]Significance levels of tests of the null hypothesis that the detrended first difference of consumption is uncorrelated with the lagged, detrended first difference of consumption and either lagged, detrended income or lagged, detrended consumption minus income. The results are grouped by the number of the lags included in the tests. The variables are defined in Section 2.

TABLE II

Results of Tests of First–Order Autocorrelations of Detrended Consumption First Differences[a]

Sample Period	c_{nd}	$c_{nd} + c_s$	c
1951.3–85.3	.260 (.070)	.237 (.072)	.256 (.078)
1952.3–85.3	.276 (.072)	.269 (.065)	.276 (.087)
1951.3–79.1	.250 (.082)	.201 (.080)	.269 (.086)

[a]Estimates of the first–order autocorrelation of the detrended first difference of consumption (with standard errors in parentheses). Column headings indicate the measure of consumption used. For variable definitions, see Section 2.

TABLE III

Results of GMM Tests of Continuous Time Random Walk Hypothesis for Consumption[a]

	Lagged, Detrended Income			Lagged, Detrended Consumption Minus Income		
Sample Period	$c, \tilde{y}$	$c_{nd} + c_s, y_d$	c_{nd}, y_d	$c, \tilde{y}$	$c_{nd} + c_s, y_d$	c_{nd}, y_d
1951.3–85.3	.008	.170	.024	.028	.210	.029
1952.3–85.3	.041	.187	.012	.118	.270	.091
1951.3–79.1	.040	.174	.135	.242	.180	.090

[a]Significance levels of tests of the joint hypothesis that the detrended first difference of consumption is uncorrelated with explanatory variables lagged 2, 3, and 4 periods and that the first–order autocorrelation of detrended consumption first differences equals 0.25. The results are grouped by the explanatory variables besides detrended consumption first differences: detrended income and detrended consumption minus income.

TABLE IV

Parameter Estimates and Test Statistics for the DLR and SLR Models

Parameters and Statistics		DLR Model Results with δ Unconstrained	DLR Model Results with δ Constrained[a]	SLR Model Results with δ Unconstrained	SLR Model Results with δ Constrained[a]
Point Estimates (and Standard Errors)	a_1	.163 (.060)	.152 (.060)	—	—
	a_2	36.12 (147.31)	27.57 (81.81)	—	—
	f	—	—	.136 (.043)	.089 (.035)
	a	—	—	12.19 (12.44)	11.75 (11.29)
	π	$.41 \times 10^{-9}$ ($.20 \times 10^{-2}$)	.0032 (.058)	.041 (.032)	.058 (.042)
	δ	.0088 (.0015)	.0098 (—)	.0078 (.0007)	.0098 (—)
	V_c[b]	$\begin{bmatrix} 484.63 & 496.29 \\ 496.29 & 720.71 \end{bmatrix}$	$\begin{bmatrix} 488.51 & 521.63 \\ 521.63 & 719.52 \end{bmatrix}$	$\begin{bmatrix} 663.74 & 317.06 \\ 317.06 & 265.22 \end{bmatrix}$	$\begin{bmatrix} 659.82 & 483.60 \\ 483.60 & 465.28 \end{bmatrix}$
Log Likelihood		−843.122	−843.249	−841.601	−842.982
Likelihood Ratio Statistics[c] (and Marginal Significance Levels for Chi-Square Distribution)	SARMA(3,4)				
	J_T	25.21 (.044)	25.76 (.058)	22.46 (.096)	25.22 (.066)
	J_T^*	23.79 (.069)	24.02 (.089)	20.95 (.138)	23.52 (.100)
	Degrees of Freedom	15	16	15	16
	SARMA(2,3)				
	J_T	22.49 (.013)	22.74 (.019)	19.44 (.035)	22.20 (.023)
	J_T^*	21.37 (.019)	21.61 (.028)	18.48 (.047)	21.10 (.032)
	Degrees of Freedom	10	11	10	11

[a]Here $\delta = r$. Throughout, $r = 0.0098$ and $\theta = 0.004568$.

[b]The variable V_c is the covariance matrix of the vector $[\eta_1(t)/\delta(a_1+\delta),\ \eta_2(t)/\delta(a_2+\delta)]$.

[c]The statistic J_T tests the overidentifying restrictions of the model against the unconstrained alternative. The corresponding adjusted statistic, J_T^*, is described in Section 4.B.

TABLE V

VAR Representations[a]

$$Q_t = A_1 Q_{t-1} + \dots + A_4 Q_{t-4} + X_t, \quad EX_t X_t{}' = \Sigma$$

	OLS Estimate	DLR Model		SLR Model	
	Parameters (and S.E.)	δ Constrained	δ Unconstrained	δ Constrained	δ Unconstrained
Coefficient	(1)	(2)	(3)	(4)	(5)
A_1	$\begin{bmatrix} 1.070 & -.136 \\ (.084) & (.132) \\ -.074 & .313 \\ (.055) & (.085) \end{bmatrix}$	$\begin{bmatrix} 1.113 & -.046 \\ .009 & .271 \end{bmatrix}$	$\begin{bmatrix} 1.126 & -.033 \\ .011 & .270 \end{bmatrix}$	$\begin{bmatrix} 1.216 & -.101 \\ -.085 & .244 \end{bmatrix}$	$\begin{bmatrix} 1.241 & -.072 \\ -.078 & .277 \end{bmatrix}$
A_2	$\begin{bmatrix} -.204 & -.206 \\ (.084) & (.130) \\ .065 & .125 \\ (.054) & (.085) \end{bmatrix}$	$\begin{bmatrix} -.297 & -.001 \\ -.013 & -.072 \end{bmatrix}$	$\begin{bmatrix} -.300 & .018 \\ -.016 & -.073 \end{bmatrix}$	$\begin{bmatrix} -.335 & .010 \\ .012 & -.099 \end{bmatrix}$	$\begin{bmatrix} -.343 & .038 \\ -.010 & -.075 \end{bmatrix}$
A_3	—	$\begin{bmatrix} .079 & -.022 \\ .006 & .020 \end{bmatrix}$	$\begin{bmatrix} .080 & -.007 \\ .008 & .019 \end{bmatrix}$	$\begin{bmatrix} .094 & -.042 \\ .004 & -.005 \end{bmatrix}$	$\begin{bmatrix} .096 & -.016 \\ .007 & .020 \end{bmatrix}$
A_4	—	$\begin{bmatrix} -.021 & -.014 \\ -.002 & -.004 \end{bmatrix}$	$\begin{bmatrix} -.021 & .003 \\ -.003 & -.005 \end{bmatrix}$	$\begin{bmatrix} -.023 & -.022 \\ .001 & -.030 \end{bmatrix}$	$\begin{bmatrix} -.024 & .006 \\ .001 & -.005 \end{bmatrix}$
Σ	$\begin{bmatrix} 217.02 & -.867 \\ -.867 & 92.74 \end{bmatrix}$	$\begin{bmatrix} 254.92 & -19.01 \\ -19.01 & 101.62 \end{bmatrix}$	$\begin{bmatrix} 255.87 & -17.93 \\ -17.93 & 101.52 \end{bmatrix}$	$\begin{bmatrix} 267.50 & -23.39 \\ -23.39 & 97.81 \end{bmatrix}$	$\begin{bmatrix} 268.25 & -22.90 \\ -22.90 & 100.13 \end{bmatrix}$

[a]Here Q_t is demeaned $[\phi^{-t}(c_t - \tilde{y}_t), \phi^{-t}\Delta c_t]'$ and X_t is the VAR innovation. Results in columns (2)–(5) are truncated VARs of indicated estimated continuous time models. These VARs were obtained by first computing a discrete time SARMA representation as indicated in footnote 8. The associated infinite ordered VAR was then found. These are what is reported in the table, with matrix coefficients on lags greater than 4 omitted.