Money in Consumption Loan Type Models: An Addendum

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1. The Cass-Okuno-Zilcha Conjecture

In the paper by Cass, Okuno, and Zilcha in this volume, we presented several examples of consumption loan type models exhibiting the nonexistence of any competitive equilibrium which is Pareto optimal. One sort of example, involving nonmonotonicity in tastes, seemed very special (see section 5.2). Evidently it depended on having just the right combination of some consumption satiation and some boundary endowments. We conjectured that such speciality wasn't essential to the intuitive proposition being advanced, that somewhat less than total satiation might unfavorably restrict potential intertemporal market transfers — even given the institution of money, in its role as a store of value, as a common means of facilitating trade between the present and the future. Rather, we proposed as a more likely reason for such speciality the particular analytic methodology being utilized, the limitation to considering only models for which competitive equilibria could be essentially characterized as the nonnegative solutions to a first-order difference equation (and thus completely represented in terms of a two-dimensional diagram).

It turns out that while our intuition was correct, our reasoning wasn't. In fact, it is now apparent that the only operative constraint was simply our lack of sufficient ingenuity.

In the next section I sketch a robust nonoptimality example; nonexistence of Pareto optimal equilibrium persists in the presence of small perturbations in both tastes and endowments. The peculiar nature of the satiation phenomena introduced in this example entails other interesting anomalies as well. Two of these additional results are briefly discussed in the final section.

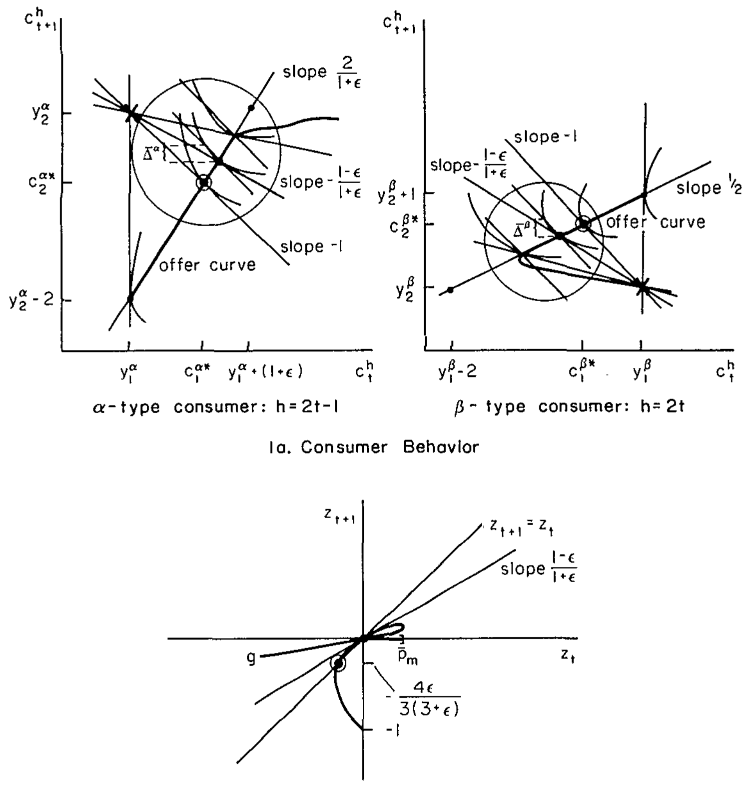
2. A Robust Nonoptimality Example

Consider the basic model described in section 2 of Cass-Okuno-Zilcha, and suppose once again that there are two consumers in every generation but the oldest, $G_0 = \{0\}$ and $G_t = \{2t-1, 2t\}$ for $t \geqq 1$, and that odd-numbered

*This note was written during my visit as a Sherman Fairchild Distinguished Scholar at the California Institute of Technology. I am very grateful to Caltech — and especially its fine group of social scientists — for providing me this excellent opportunity to conduct uninterrupted research in such a congenial environment.

consumers, $h = 2t-1$ for $t \geqq 1$, are of α-type; even-numbered, $h = 2t$ for $t \geqq 1$, of β-type. Their respective tastes and endowments are assumed to be as specified in Figure 1a. The critical general features of this specification are that both α- and β-type consumers can become satiated in second-period consumption — though at least one type can never become satiated in first-period consumption — while the α- but not the β-type consumer is relatively overburdened with second-period income. Figure 1b exploits the more specific features introduced in order to simplify exposition, especially the particular linear structure of the offer curves of both types of consumers at all except very low real rates of return. The figure makes plain that, while there

Figure 1
Nonoptimality Due to Nonmonotonicity:
An Example With Potential Satiation
in Second-Period Consumption



1a. Consumer Behavior

1b. Dynamical System

are a plethora of both barter and monetary equilibria in this example, nevertheless every competitive equilibrium must exhibit real rates of return which satisfy the uniform bound

$$\frac{\partial U^h(c^h)}{\partial c_t^h} \bigg/ \frac{\partial U^h(c^h)}{\partial c_{t+1}^h} = p_t/p_{t+1} = z_{t+1}/z_t \leqq (1-\epsilon)/(1+\epsilon) < 1$$

for $h = 2t-1, 2t$ and $t \geqq 1$.

Hence, it is easily seen (referring to the encircled portions of Figure 1a and appealing to the specific characteristics of the geometric construction briefly remarked in the following paragraph) that every competitive allocation is dominated by a corresponding feasible allocation which is identical except for a sequence of sufficiently small one-for-one forward transfers between only α-type consumers

$$0 < -\Delta c_t^{2t-1} = \Delta c_{t+1}^{2t-1} = \Delta c^\alpha < \bar{\Delta}^\alpha \text{ for } t \geqq 1$$

(or alternatively, only β-type consumers

$$0 < -\Delta c_t^{2t} = \Delta c_{t+1}^{2t} = \Delta c^\beta < \bar{\Delta}^\beta \text{ for } t \geqq 1).$$

That this example is legitimate and, more importantly, that its welfare significance is invariant to sufficiently small perturbations in both tastes and endowments should become self-evident from close examination of Figure 2. This figure contains directions for constructing a well-behaved utility function consistent with my specification of the α-type consumer; a similar procedure can be employed to justify my specification of the β-type consumer. Since both constructs are very closely patterned after that already analyzed at great length in Cass-Okuno-Zilcha (see Appendix section A2), I omit further elaboration here.

3. Additional Difficulties Associated With Satiation Phenomena

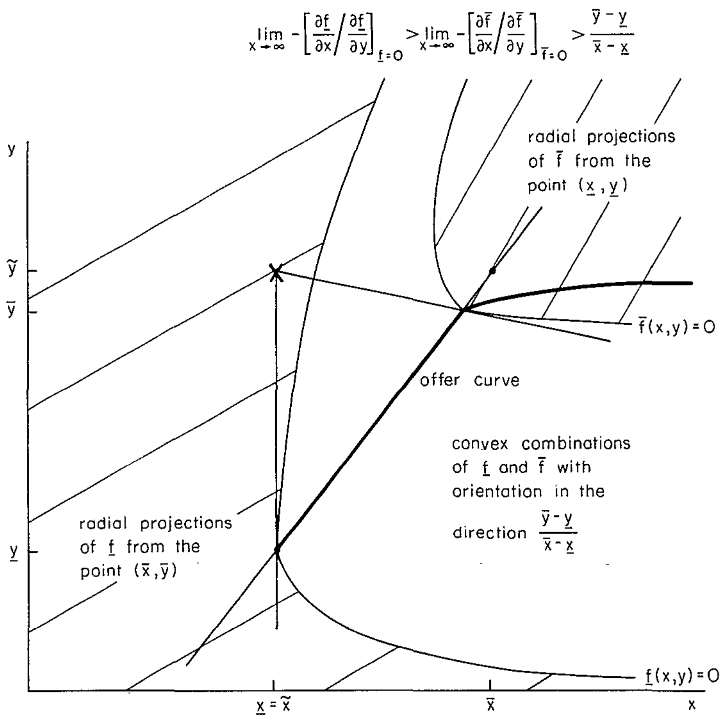
3.1. *Nonexistence of Competitive Equilibrium*

Figure 3 displays a technically minor modification of the preceding example which entails a substantively major conclusion. By merely shifting portions of both consumers' indifference maps so that the α-type (respectively, β-type) consumer's offer curve is considerably less (more) steeply sloped at negative real rates of return near zero, competitive equilibrium ceases to exist.

At first blush this result is somewhat surprising. All consumers' tastes and endowments continue to enjoy the same regularity properties as before. Upon closer inspection, however, the source of the difficulty is easily discovered. The peculiar second-period nature of satiation has now combined with the intrinsic one-directional nature of time in just such a manner that there is simply not enough structural interrelationship between generations to admit trade—even trade only among members of each generation—through the market mechanism. Deficiency of this sort is well-known to create difficulties for the existence of competitive equilibrium in the standard Arrow-Debreu model. [See, in particular, the illuminating seminal analysis of McKenzie (1959)†] This version of the example suggests that in the intertem-

†Author names and years refer to the works listed at the end of this book.

Figure 2
Construction of an Offer Curve Exhibiting Perverse Behavior at Extreme Relative Prices



poral context these difficulties should not be peremptorily dismissed as mere curiosities. It also hints at a commonality between the circumstances giving rise to nonoptimality and those giving rise to nonexistence which should be thoroughly investigated.

3.2. *Ineffectiveness of Social Security*

Subject to well-known standard qualifications, the second basic theorem of welfare economics, that every Pareto optimal allocation is a competitive equilibrium (given an appropriate initial distribution of endowments), obtains in consumption loan type models. [See, in particular, the even more comprehensive analysis of McFadden, Majumdar, and Mitra (forthcoming).] In the present example (in either its original or modified form), for instance, any redistribution of endowments according to the scheme

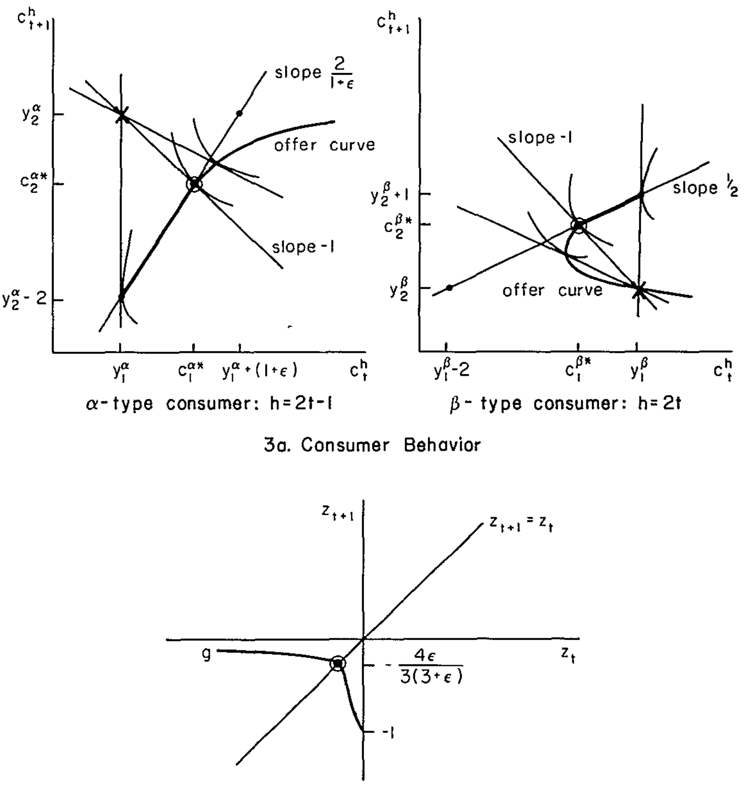
$$y^{h\delta} = \begin{cases} y^0 - \dfrac{4\epsilon}{3(3+\epsilon)} & \text{for } h = 0 \\ c^{\alpha *} + (\delta,-\delta) & \text{for } h = 2t-1,\ t \geqq 1 \\ c^{\beta *} + (-\delta,\delta) & \text{otherwise} \end{cases}$$

permits attaining a barter equilibrium which is Pareto optimal (supported by prices $p_t = 1$ for $t \geqq 1$ and yielding the allocation

$$c^h = \begin{cases} y^0 - \dfrac{4\epsilon}{3(3+\epsilon)} & \text{for } h = 0 \\ c^{\alpha *} & \text{for } h = 2t-1,\ t \geqq 1 \\ c^{\beta *} & \text{otherwise}). \end{cases}$$

The central message of this example, of course, is that the historical advent of money as a store of value may not be an adequate proxy for such a redistribution of endowments. Thus there remains at least partly unanswered an extremely interesting question: What are the simplest extra-market in-

Figure 3
Version of the Example With No Competitive Equilibrium
(Barter or Monetary)



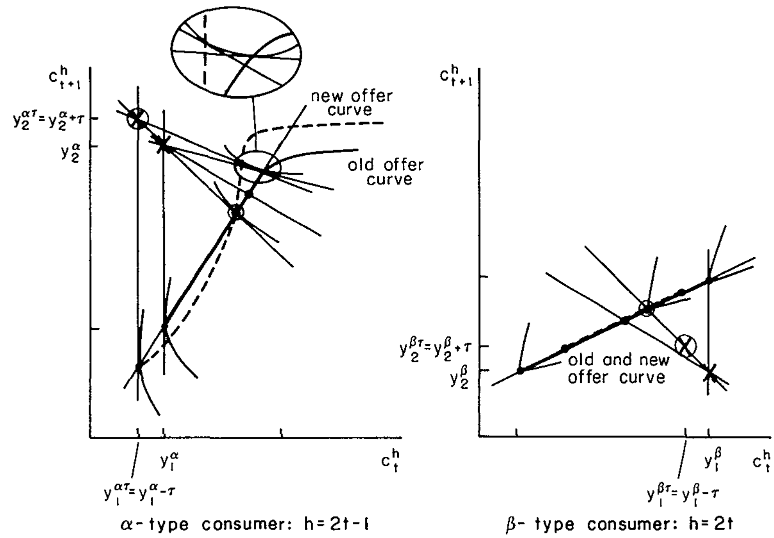
3a. Consumer Behavior

3b. Dynamical System

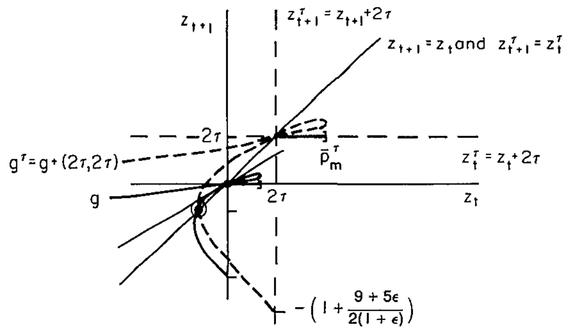
stitutional arrangements — perhaps involving some direct redistribution of endowments — which will permit attaining Pareto optimality in a wide class of intertemporal market environments?

I hazard the opinion, based primarily on extensive analysis of consumption loan type models embodying both variety of commodities and diversity of consumers, that a minimal qualification for any such arrangement will be a large degree of flexibility in confronting heterogeneity across agents.

Figure 4
Version of the Example With a Simple Social Security System Which Will Not Permit Attaining Pareto Optimality



4a. Consumer Behavior



4b. Dynamical System

This view is suggestively supported (but by no means precisely or conclusively demonstrated) by considering, for instance, the scope of a perpetual per capita transfer τ from young to old — or a simple social security system — in the present example. It is a fairly straightforward matter to establish that whether or not such uniform social transfers are potent enough to permit attaining Pareto optimality through additional market transfers is problematical and ultimately depends on whether or not they conceivably admit barter equilibrium at some nonnegative real rate of return. Figure 4 presents a polar version of the example in which a simple social security system is necessarily ineffective. The figure is more or less self-explanatory once it is noted that any shift from first- to second-period income essentially results in some distorted rotation of the reflected generational offer curve (still defined relative to original endowments) around its intersection with the 45° line in the negative quadrant.

Clearly, this counterexample depends on the particular structure of the distortion displayed in Figure 4b, which in turn depends on the particular character of the behavior displayed in Figure 4a — especially the relative sensitivity of the α-type consumer (and the absolute insensitivity of the β-type consumer) to shifts from first- to second-period income. The opposite polar version of the present example, reversing these sensitivities, yields a model in which a simple social security system is potentially effective.